REVIEW PROBLEM
3-46. Given the periodic waveform shown:

(a) What is the value of $a_0$ in the sine-cosine Fourier series? Why?
(b) What are the values of the $b_k$ coefficients in the sine-cosine Fourier series? Why?
(c) Are the coefficients in the complex exponential Fourier series real, imaginary, or complex? Why?
(d) Does this waveform have half-wave odd symmetry?
With $s = \sigma + jo$, (4-11) can be written as

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$  \hspace{1cm} (4-2)$$

where the script letter $\mathcal{L}$ denotes the operation of obtaining the Laplace transform of $x(t)$ defined by (4-2). Since the lower limit of the integral is zero, (4-2) is referred to as the single-sided Laplace transformation.

Use of the Laplace transform for system analysis results in several advantages. Among these are:

1. The solution of differential equations progresses systematically and involves only algebraic manipulations.
2. The method provides a method of expressing physical quantities in an alternate form.
3. Conditions are automatically included in the solution of the equation.

Not every signal possesses a Laplace transform. An example is $\exp(-at)$, $a > 0$, which grows faster than $\exp(-st)$ decays. Thus the Laplace transform integral (4-2) does not converge for this signal. We will show convergence properties of Laplace transforms more fully in the following section.

The operation that changes $X(s)$ back to $x(t)$ is referred to as the inverse Laplace transform and is symbolized by $\mathcal{L}^{-1}$. To obtain $x(t)$ in terms of $X(s) = X(s) + joX$, we observe from the inverse Fourier transforms of (4-1) that

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}ds$$  \hspace{1cm} (4-3a)$$

Multiplying both sides by $e^t$ and summing over a constant, we obtain the inverse integral,

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}ds$$  \hspace{1cm} (4-3b)$$

where the change of variable $s = \sigma + jo$ and $\Delta = j\omega$ has been used. With $\sigma, \omega = \omega \in (4-3b)$, the limits for (4-3b) are clearly $\sigma, \omega = \infty$. Although not easy to show rigorously, (4-3b) may be specialized to a complex inversion integral where $\sigma = \omega = \infty$ in a complex variable with a real part that may vary along with its imaginary part as it is decreased in more detail in Section 4-5, where the extent of contour integration is briefly examined for the purpose of evaluating inverse Laplace transforms.

For the most part, contour integration can be avoided when finding inverse Laplace transforms in systems analysis problems simply by making use of a table of Laplace transform pairs. We now begin the construction of such a table and illustrate the evaluation of (4-2) for several simple signals.

---

**4.4 EXAMPLES OF EVALUATING LAPLACE TRANSFORMS**

As an example of evaluating (4-2), let $x(t) = 1$. Then

$$X(s) = \int_{0}^{\infty} e^{-st}dt$$

$$= \left[ \frac{-e^{-st}}{s} \right]_{0}^{\infty} = \frac{1}{s}$$

Clearly, unless $\sigma = \Re(s) > 0$, the limit as $s \to 0$ does not exist since $\int_{0}^{\infty} e^{-st}dt = 1$ and $\int_{0}^{\infty} e^{-st}dt = 0$ when $\sigma > 0$. Thus $\sigma > 0$ grows without bound if $\sigma < 0$. Requiring that $\Re(s) > 0$, we obtain

$$\int_{0}^{\infty} e^{-st}dt > \int_{0}^{\infty} e^{-st}dt$$

(4-4)

Because the lower limit of (4-2) is zero, values of $x(t)$ for $t < 0$ have no effect on $X(s)$. We see that $x(t) = 1$ and $x(t) = \sin(\omega t)$ are the same single-sided Laplace transforms. If we consider signals that are nonzero only for $\sigma > 0$, it is possible to treat this as a Fourier integral. This is reasonable for our purposes because all signals must start sometime. We may choose the starting time to be consistent as $t = 0$.

Some discussion is in order about the convergence of the Laplace transform integral. In obtaining the Laplace transform of $\sin(\omega t)$, it was necessary to extend $x(t)$ by $\sigma > 0$ or more general for the integral to exist. Recall that an integral with one or both of the limits extended is referred to as an improper integral. An improper integral of the form

$$\int_{t}^{\infty} f(t)dt$$

is said to be convergent if the limit on the right-hand side exists. The integral converges absolutely if and only if

$$\int_{t}^{\infty} |f(t)|dt < \infty$$

The Laplace transform integral of a signal $x(t)$ can be shown to converge absolutely for all

$$\sigma = \Re(s) > \epsilon$$

(4-5)

for any positive real number $\epsilon$ and some real constant $K$, if and only if $\int_{0}^{\infty} |x(t)|dt < \infty$. An introduction to this topic is given in Section 4-5.

---

*The property of a signal which makes it possible to evaluate the Fourier transform of the signal as defined in Section 4-4 is said to be $\omega$-band limited. That is, the spectrum of the signal is contained within the band $-\pi \omega \leq \omega \leq \pi \omega$. This is the contour of the spectrum of the system differential equation with the system characteristic zero.*
where $A$ and $c$ are appropriately chosen real constants. Such a signal is said to be of exponential order. The smallest possible value of $c$ is called the order of absolute convergence. A proof of this theorem is given in Appendix C.

Applying this theorem to $s(t) = m(t)$, we see that $A = 1$ and $c = 0$ are appropriate choices, and the Laplace transform of $s(t)$ therefore converges absolutely for $Re(s) > 0$.

By applying the theorem to $s(t) = e^{at}$, with $L > 0$, $A = 1$, and $c = 0$, we know that the signal also has a Laplace transform which, in fact, can be shown to be $m(s)/s^L$ for $Re(s) > 0$. Since the importance of the role of $L$ in this example,

Consider next the Laplace transform of the signal

$$ \mathcal{L}\{m(t)\} = \frac{1}{s - a} $$

where $a$ may be complex. By definition, the Laplace transform is

$$ \mathcal{L}\{m(t)\} = \int_0^\infty e^{-st}m(t)\,dt $$

or

$$ X(s) = \frac{1}{s - a} $$

The absence of absolute convergence, or simply absence of convergence, in this case is

$$ e = \Re(a) $$

At a final example, we obtain the Laplace transforms of $e^t$. By definition of the Laplace transform, it is

$$ \mathcal{L}\{e^t\} = \int_0^\infty e^{-st}e^t\,dt $$

whereupon we are faced with a dilemma. Since the unit impulse function occurs at $t = 0$, we integrate through half of $t$, none of it, or all of it. We will assume that this book that the lower limit of the Laplace transform is $s = 0$.

Thus (4-1) evaluates to (4-2) $X(s) = 1$. The case of $s = 0$ makes no difference, that is, the region of convergence is the entire $s$ plane.

A final word about convergence. The notion of convergence for the signals $m(t)$ and $e^{at}$ are shown in Figure 4-1a together with the regions of absolute convergence of their respective Laplace transform integrals. For any value of $s$ lying in the region of convergence, the respective Laplace transform is finite. It follows that any singularities of the Laplace transform of a signal must lie to the left of the region of absolute convergence. The Laplace transform of the unit step and decaying exponential signals, reveal a function of the complex variable $s$, each have a single singularity which is located to the right of the region of absolute convergence.
TABLE 4-1: Table of Laplace Transforms

<table>
<thead>
<tr>
<th>Signal</th>
<th>Laplace Transform</th>
<th>Region of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s ) 1</td>
<td>( \frac{1}{s} )</td>
<td>( s &gt; 0 )</td>
</tr>
<tr>
<td>( s^m )</td>
<td>( \frac{s^m}{s^{m+1}} )</td>
<td>( s &gt; 0 )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{1}{s+a} )</td>
<td>( s &gt; -a )</td>
</tr>
</tbody>
</table>

The integral of (4-3) is carried along any line to the right of the singularities of \( F(s) \). If this line can be chosen as the axis of \( a = 0 \), it must follow from (4-1) and (4-2) that the ordinary Fourier transform of \( e^{-at} \) exists and can be obtained from \( \mathcal{L}(s) \) by substituting \( s = j\omega \).

The next Laplace transform pair just derived are summarized in Table 4-1 together with their absolute of absolute convergence. We will dispense with specifying the absolute value of the convergence region of Laplace transforms since it can be shown that there is no ambiguity in the inverse single-sided Laplace transforms, even with the region of convergence unspecified. This is not true of the double-sided Laplace transform for which it is necessary to specify the region of convergence together with the Laplace transform in order to uniquely specify the inverse Laplace transform. Since we are not considering the four-sided Laplace transform, specifications of the absolute values of convergence are not necessary.

We now will extend Table 4-1, but to avoid the labor of carrying out integrations, we will give several Laplace transform pairs and one examples to extend it. Because the Fourier and Laplace transforms are both linear integral transforms, many of the theorems to be used will be self-evident from our consideration of the Fourier transform.

**4-3 SOME LAPLACE TRANSFORM THEOREMS**

**Theorem 1: Linearity:** Since (4-1) is an integral in which \( f(t) \) appears linearly, it follows, for two constants \( a \) and \( b \) which may be complex, that

\[
\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]
\]

where \( \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt \) and \( \mathcal{L}[g(t)] = \int_0^\infty e^{-st}g(t)\,dt \). The linearity theorem will be made use of in the Laplace transformation of differential equation. We illustrate its use for extending Table 4-1 by an example.

**EXAMPLE 4-1**

The Laplace transform of \( \sin kt \) may be obtained by expressing it in exponential form, applying the linearity theorem, and using the transform pair

\[
\mathcal{L}[e^{\alpha t}f(t)] = \int_0^\infty e^{-st}e^{\alpha t}f(t)\,dt = \int_0^\infty e^{(s-\alpha)t}f(t)\,dt
\]

with \( \alpha = \pm j\omega \), which gives

\[
\mathcal{L}[e^{\pm j\omega t}] = \int_0^\infty e^{-st}e^{\pm j\omega t}\,dt = \int_0^\infty e^{(s-\omega)t}f(t)\,dt = \frac{1}{s\\pm j\omega}
\]

In a similar manner, the Laplace transform of \( \sin \omega t \) can be obtained as follows:

\[
\mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st}e^{j\omega t}\,dt = \frac{1}{s^2 + \omega^2}
\]

**Theorem 2: Transform of Derivatives:** From (4-2) the Laplace transform of \( \frac{d}{dt}f(t) \) is

\[
\mathcal{L}\left[ \frac{d}{dt}f(t) \right] = -s\mathcal{L}[f(t)] + f(0+)
\]

Which may be integrated by parts by letting \( u = e^{-st} \), and \( dv = f(t)\,dt \) in the equation

\[
\int_0^\infty f(t)\,dt = \frac{\partial}{\partial s}\int_0^\infty e^{-st}f(t)\,dt = \frac{\partial}{\partial s}\mathcal{L}[f(t)]
\]

Then \( dv = -e^{-st}\,dt \) and \( v = \frac{1}{s} \), so that (4-10) becomes

\[
\mathcal{L}\left[ \frac{d}{dt}f(t) \right] = -s\mathcal{L}[f(t)] + f(0+)
\]

provided that \( \lim_{t \to 0} f(t) = 0 \), where we recall that we have agreed to use \( f(0) \) for the lower limit.

To find \( \mathcal{L}\left[ \frac{d^n}{dt^n}f(t) \right] \) we use

\[
\frac{d^n}{dt^n}f(t) = \frac{d^n}{dt^n} \mathcal{L}[f(t)] = \frac{d^n}{dt^n} \int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-st} \frac{d^n}{dt^n}f(t)\,dt
\]

**4-3 SOME LAPLACE TRANSFORM THEOREMS**

174
which results in

\[ V \left. \frac{d^2 x(t)}{dt^2} \right|_{t=0} = \int_0^\infty \left( e^{-st} - 1 \right) x(t) dt \]

\[ = x(0) - 0 + \left. \frac{dx(t)}{dt} \right|_{t=0} \]

\[ = x(0) + \left. \frac{dx(t)}{dt} \right|_{t=0} - x(0) \]

Using integration, we may show that

\[ \int_0^\infty \left( e^{-st} - 1 \right) x(t) dt = \int_0^\infty e^{-st} x(t) dt - x(0) \]

where \( x(t) \) is the 4th derivative of \( x(t) \) evaluated at \( t = 0 \). The use of (4-12) will be illustrated with an example.

**EXAMPLE 4-2**

Consider the circuit shown in Figure 4-3, where the switch is operated from 0 to 1 at \( t = 0 \). Find the current through the inductor as a function of time.

**Solution:** The inductor current obeys the differential equation

\[ \frac{dx(t)}{dt} + 2x(t) = 4, \quad t > 0 \]

\[ 0, \quad t < 0 \]

Taking the Laplace transform of both sides starting at \( t = 0 \), we obtain

\[ sX(s) - x(0) + 2X(s) = \frac{4}{s} \]

To proceed further, we require \( x(0) \). Assuming that the circuit was in steady state for \( t < 0 \), we see from the circuit diagram that

\[ x(0) = \frac{1}{2} \]

Thus

\[ x(t) = \frac{1}{s} \left( \frac{4}{s} + 2x(0) \right) \]

or

\[ x(t) = \frac{2}{s} + 2 \]

**FIGURE 4-2.** Circuit for the illustration of Laplace transform solution techniques.

Using (4-5) the current through the inductor is

\[ I(t) = \begin{cases} \frac{2}{s}, & t > 0 \\ 0, & t \leq 0 \end{cases} \]

**Theorem 3: Laplace Transform of an Integral**

The Laplace transform of

\[ \int_{0}^{t} x(t) \, dt \]

is

\[ \frac{X(s)}{s} \]

where \( x(s) \) is the Laplace transform of \( x(t) \) evaluated at \( t = 0 \). This theorem is proved by using the formula for integration by parts. With

\[ a = x(t) \]

\[ b = 1 \]

\[ c = s \]

\[ d = x(t) \]

the Laplace transform of (4-14) becomes

\[ \int \left[ \frac{1}{s} X(s) \right] \, dt = \left. \frac{1}{s} x(t) \right|_{0}^{t} - \int_{0}^{t} \frac{1}{s} x(t) \, dt \]

which gives (4-15) when the limits are substituted provided that

\[ \lim_{s \to 0} \frac{1}{s} x(t) \, dt = 0 \]

**EXAMPLE 4-3**

This proof the integration theorem will be illustrated by finding an expression for the Laplace transform of the current in the circuit of Figure 4-3. Kirchhoff's voltage law results in the loop equation

\[ \frac{2}{s} X(s) + \frac{1}{s} x(t) + \int_{0}^{t} x(t) \, dt = \frac{x(t)}{s} \]

4.3 SOME LAPLACE TRANSFORM THEOREMS
where the voltage-current relationships for each element have been substituted. Application of the linearity, differentiation, and integration theorems yields

\[ i(t) = \frac{1}{L} \int_0^t v(t) \, dt \]

where \( i(0^+) = 0 \) because the switch is open prior to \( t = 0 \) and

\[ \frac{v(0^+)}{C} \int_0^t \left( \frac{v(t)}{C} dt \right) = \frac{v(0^+)}{C} \]

is the voltage across the capacitor at \( t = 0^+ \). Solving for \( v(t) \), we obtain

\[ v(t) = \frac{L}{C} \frac{d}{dt} \left[ i(t) \right] + \frac{v(0^+)}{C} \]

The inversion of \( i(t) \) in response to particular inputs will be presented until the next section.

**Theorem 4: Complex Frequency Shift (S-Shift) Theorem**

The Laplace transform of

\[ e^{\sigma t} y(t) = \mathcal{L}\{ e^{\sigma t} y(t) \} \]

is

\[ \mathcal{L}\{ e^{\sigma t} y(t) \} = sY(s) - \sigma Y(s) \]

where \( Y(s) = \mathcal{L}\{ y(t) \} \).

The theorem is proved by substituting (4.26) into the definition of the Laplace transform and noting that the integral is the Laplace transform of \( e^{\sigma t} y(t) \) with the variable \( s > \sigma \). The proof is left to the reader at the end of the chapter. The s-shifting theorem is useful for extending the table of Laplace transforms. For example, application of the \( s \)-shift theorem to (4.26) shows that

\[ \mathcal{L}\{ e^{\sigma t} \sin \omega t \} = \frac{s^2 \omega^2 + \sigma^2}{(s^2 + \omega^2)^2} \]

while application to (4.26) results in the Laplace transform pair

\[ \mathcal{L}\{ e^{\sigma t} \sin \omega t \} = \frac{\sigma}{s^2 + \omega^2} \]

EXAMPLE 4-4

Consider the function of \( s \),

\[ X(s) = \frac{s + 8}{s^2 + 6s + 13} \]

To find the corresponding function of time, we write it as

\[ X(s) = \frac{a}{s + 3} \frac{b}{s + 3 + 4} \]

Applying the transform pairs (4.23) and (4.24), we obtain

\[ x(t) = e^{-3t} \left( 2e^{4t} \right), \quad t > 0 \]

**Theorem 5: Delay Theorem**

If the Laplace transform of \( x(t) \) is \( X(s) \), then

\[ \mathcal{L}\{ x(t - \tau) \} = e^{-\tau s} X(s) \quad \text{for } \tau > 0 \]

The proof follows easily by using the definition of the Laplace transform (4.21) with \( x(t) = \delta(t - \tau) = 1 \) for \( t \leq \tau \) and 0 otherwise. Since \( x(t) = 0 \) for \( t < \tau \), we obtain

\[ \mathcal{L}\{ x(t - \tau) \} = \mathcal{L}\{ \delta(t - \tau) \} = e^{-\tau s} \]

Letting \( e^{-\tau s} \) be in the integrand, this becomes

\[ \mathcal{L}\{ x(t - \tau) \} = \int_{-\infty}^{\tau} e^{-\tau s} e^{-\tau s} \, dt = \mathcal{L}\{ x(t) \} \]

which proves the theorem.

We note that the step function \( x(t - \tau) = 1 \) is necessary in (4.26) to give the proper lower limit on the Laplace transform. Equation (4.26) does not hold if \( \tau < 0 \) since the single-sided Laplace transform will not include the portion of \( x(t) = \delta(t - \tau) \), that exists for \( t < 0 \).

EXAMPLE 4-5

The Laplace transform of a square wave beginning at \( t = 0 \) is

\[ x(t) = \frac{1}{2} + \frac{1}{2} \left( e^{-\tau s} - 1 \right) \]

Using the series

\[ \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots \quad |x| < 1, \]

THE LAPLACE TRANSFORMATION: APPLICATIONS TO SYSTEMS ANALYSIS 179

4.3 SOME LAPLACE TRANSFORM THEOREMS
we obtain

\[ x(t) \ast y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) \, d\tau \]

where \( x(t) \) is a square wave with amplitude \( \pm 1 \) and period \( T_x \).

**Theorem 6: Laplace Transform of the Convolution of Two Signals**

Given two signals, \( x(t) \) and \( y(t) \), which are zero for \( t < 0 \), their convolution is

\[
x(t) \ast y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) \, d\tau
\]

\[
= \int_{0}^{\infty} x(\tau) y(t - \tau) \, d\tau
\]

The last expression follows by virtue of \( x(t) = 0 \) for \( t < 0 \) or \( y(t) = 0 \), \( \lambda > t \). By definition of the Laplace transform, we have

\[
X(\lambda) = \mathcal{L}\{x(t)\} = \int_{0}^{\infty} e^{-\lambda t} x(t) \, dt
\]

Integrating by parts, we have

\[
X(\lambda) = \int_{0}^{\infty} e^{-\lambda t} x(t) \, dt = x(0) + \lambda X(\lambda)
\]

where the change of variable \( \eta = t - \lambda \) has been made. We recognize the integral as the Laplace transform of \( x(\lambda - t) \) and the same integral as the Laplace transform of \( y(t) \). Thus, the Laplace transform of the convolution of two signals is the product of their respective Laplace transforms, and the region of absolute convergence consists of the intersection of the regions for \( X(\lambda) \) and \( Y(\lambda) \).

**Theorem 7: Laplace Transform of a Product**

The Laplace transform of a product of two signals can be expressed as an integral in the complex plane. We will not use it in this chapter, but it is given in the table of Laplace transforms for future reference.

**Theorem 8: Initial Value Theorem**

The Laplace transform of the derivative of a signal has been shown to be

\[
\mathcal{L}\{\frac{dx}{dt}\} = sX(s) - x(0)
\]

If \( x(t) \) is continuous at \( t = 0 \), \( x(0) \) does not contain an impulse at \( t = 0 \).

Therefore, as \( s \to \infty \) with \( s > 0 \), the absolute of convergence, the integral variables, giving

\[
\lim_{s \to \infty} X(\lambda) = sX(s)
\]

If \( x(t) \) is discontinuous at \( t = 0 \), then \( x(0) \) contains an impulse \( \delta(t) \).

\[
\lim_{s \to \infty} \int_{-\infty}^{\infty} \frac{dx}{dt} e^{-st} \, dt = \lim_{s \to \infty} sX(s) - x(0)
\]

Thus

\[
x(0) = \lim_{s \to \infty} X(s) - \lim_{s \to \infty} sX(s)
\]

provided that \( sX(s) \) exists. If the lower limit on the Laplace transform had been taken as \( 0 \), the impulse would not have been included and we would have again obtained (4-30). Thus, \( x(0) \) has been provided that it exists, always given the initial value of \( x(t) \) as \( x(0) \) regardless of the lower limit used on the Laplace transform integral.

**EXAMPLE 4-6**

The initial value of \( \exp(-\lambda x) \cos s \omega t \) is given by

\[
\lim_{s \to \infty} \frac{1}{s + \omega^2} = 1
\]

where the transform pair has been used.

The initial value of \( \exp(-\lambda x) \sin s \omega t \) is given by

\[
\lim_{s \to \infty} \frac{s}{s^2 + \omega^2} = \omega
\]

where the transform pair has been used.

The initial value of \( \exp(-\lambda x) \) has been used.

The initial value of the entire function \( \mathcal{L}\{e^{-\lambda t} + 10\} \) cannot be found since \( \lim_{s \to \infty} e^{-\lambda t} = 0 \) does not exist. The reason is that

\[
\lim_{s \to \infty} \frac{1}{s + 10} = \frac{1}{10}
\]

has the inverse Laplace transform \( x(t) = \frac{1}{10} e^{-10t} x(t) \), which has an impulse at \( t = 0 \).

**Theorem 9: Final Value Theorem**

If \( x(t) \) and \( \mathcal{L}\{x(t)\} \) are Laplace transformable, then

\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} X(s)
\]
provided that \( \mathfrak{m}(w, w') \) exists, which in the case of \( \mathfrak{L}(s)^2 \) has no poles on the \( j w \)-axis or in the right-half plane. The proof of this theorem is left to the problems.

**EXAMPLE 4.7**

The final value of a unit step is given by

\[
\lim_{t \to \infty} u(t) = 1
\]

The final value of \( e^t \) is obtained from

\[
\lim_{t \to \infty} e^t = \infty
\]

**Theorem 10: The Scaling Theorem**

The Laplace transform of \( s^n u(t) \), where \( n \) is a positive constant, is \( e^{-st} u(t) \). This is the generalization of the scalar change theorem of Fourier transforms to the complex exponential. Note that since we are dealing with the single-sided Laplace transform, \( n \) must be positive. If \( n = -1 \), for example, a time derivative in the Laplace transformed would be reversed and exist for \( t < 0 \). The resulting negative-time portion would then not be included in the Laplace transform integral.

The theorems and transform pairs that have been developed so far are collected for easy reference in Tables 4-2 and 4-3, respectively.

### INVERSION OF RATIONAL FUNCTIONS

The examples of Section 4.3 illustrate that functions involving the ratio of two polynomials \( r \), called rational functions of \( s \), are commonly occurring Laplace transforms. Indeed, this will be the form of the Laplace transform obtained when considering any small, linear, time-invariant system with a forcing function that is a power of \( e \) exponential in \( t \), a polynomial, or a combination of these.

The techniques that we shall develop for inversion of rational functions of \( s \) are valid only for proper rational functions, that is, functions for which the numerator polynomial is of degree less than the degree of the denominator polynomial. For cases where \( \theta(s) \) is not true, it is very simple to obtain a proper rational function through use of long division. For example, consider the nonproper rational function of \( s \)

\[
Z(s) = \frac{s + 2}{s + 1}
\]

This can be written as

\[
Z(s) = 1 + \frac{1}{s + 1}
\]

by use of long division. We may easily apply the Laplace transform pairs of Table 4-3 to obtain its inverse Laplace Transform. The important point is that the numerator term is a proper rational function.

To illustrate techniques for the inverse Laplace transformation of proper rational functions, we consider \( (4.20) \) for several special cases. To simplify the notation, we consider the outputs across the resistor in Figure 4.2a to be the system output \( z(t) \), and denote its Laplace transform in \( Y(s) \). Also, let

\[
\mathfrak{E}(s) = \frac{R}{L} \quad \text{(4.32)}
\]

and

\[
\theta(s) = \frac{1}{LC} \quad \text{(4.33)}
\]

Thus (4.20) becomes

\[
\mathfrak{K}(s) = \frac{\mathfrak{K}(s)}{1 + \frac{1}{LC}} = \frac{\mathfrak{K}(s)}{1 + \frac{1}{LC}}
\]

\[
(4.34)
\]
### TABLE 4-3  Extended Table of Laplace Transforms

<table>
<thead>
<tr>
<th>Signal</th>
<th>Laplace Transform</th>
<th>Comment or Derivation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 / s^n$</td>
<td>$1 / s^n$</td>
<td>Direct evaluation of $1 / s^n$ (1-25)</td>
</tr>
<tr>
<td>$1 / s^2$</td>
<td>$1 / s^2$</td>
<td>Direct evaluation</td>
</tr>
<tr>
<td>$c s^2 + d = c / s^2$</td>
<td>$c s^2 + d = c / s^2$</td>
<td>Differentiation applies to part 3, Table 4-1</td>
</tr>
<tr>
<td>$\sin (\omega t)$</td>
<td>$\omega / s^2 + \omega_0^2$</td>
<td>Example 4-1</td>
</tr>
<tr>
<td>$\cos (\omega t)$</td>
<td>$\omega_0 / s^2 + \omega_0^2$</td>
<td>Example 4-1</td>
</tr>
<tr>
<td>$e^{at} \sin (\omega t)$</td>
<td>$\omega / (s - a)^2 + \omega_0^2$</td>
<td>$-a$ and part 4</td>
</tr>
<tr>
<td>$e^{at} \cos (\omega t)$</td>
<td>$\omega_0 / (s - a)^2 + \omega_0^2$</td>
<td>$-a$ and part 5</td>
</tr>
<tr>
<td>Square wave</td>
<td>$2e^{-t / 2} - t / 2 - 2e^{-t / 2}$</td>
<td>Example 4-5</td>
</tr>
<tr>
<td>$\sin (\omega t) / s$</td>
<td>$\omega / s^2 + \omega_0^2$</td>
<td>Example 4-12 (Part 1, convolution)</td>
</tr>
<tr>
<td>$\cos (\omega t) / s$</td>
<td>$\omega_0 / s^2 + \omega_0^2$</td>
<td>Part 4 and convolution</td>
</tr>
<tr>
<td>$\exp (-at) \sin (\omega t)$</td>
<td>$-a / s^2 + \omega_0^2$</td>
<td>$-a$ and part 10</td>
</tr>
<tr>
<td>$\exp (-at) \cos (\omega t)$</td>
<td>$-a / s^2 + \omega_0^2$</td>
<td>$-a$ and part 9</td>
</tr>
</tbody>
</table>

We refer to $\xi$ and $\omega_0$ as the damping ratio and natural frequency, respectively. The roots of the denominator of (4-34) are given by:

$$
\xi^2 = 1 - \left( \frac{\omega_0}{\omega} \right)^2
$$

(4-35)

Thus, if $\xi > 1$, the roots are real and distinct; if $\xi < 1$, the roots are complex conjugates; and if $\xi = 1$, they are real and equal.

---

**EXAMPLE 4-8**

Let the input to the circuit shown in Figure 4-3 be a unit step with Laplace transform $U(s) = 1/s$. Also, assume that $\xi = 0$, $\omega_0 = 10$, and $s = 10$. Thus (4-34) becomes:

$$
X(s) = \frac{1}{s + 10} + 10
$$

Checking Table 4-3, we see that the required form for (4-36) does not appear. However, (4-36) may be expanded in partial fractions as:

$$
\frac{10}{(s + 2)(s + 8)} = \frac{A}{s + 2} + \frac{B}{s + 8}
$$

(4-37)

We have three unknowns: we can follow in determining the unknown coefficients, $A$ and $B$.

1. **Common Denominator:** Placing each factor in the right-hand side of (4-37) over a common denominator results in:

$$
10 = (s + 8)(A) + (s + 2)(B)
$$

or

$$
10 = (A + B)s + (8A + 2B)
$$

Setting coefficients of the powers of $s$ on equal basis on each side of this equation, we obtain the simultaneous equations:

$$
A = 0
$$

and

$$
B = 10
$$

The first of these equations gives $\xi = -a$, which, when substituted into the second, yields:

$$
K = 10
$$

or $\xi = -1$, and $A = 0$. Thus:

$$
Y(s) = \frac{5}{s^2 + 2s + 10}
$$

Using the linearity theorem and transform pair 3, we find $y(t)$ to be:

$$
y(t) = \frac{5}{(s^2 + 2s + 10)} e^{at} dt
$$

(4-38)

2. **Substituting Specific Values of $s$:** Since (4-37) is an identity for any $s$, we may assign two specific values for $s$ and $B$ by substituting two convenient values of $s$ (whichever are equal to either root of the denominator).

For example, substituting $s = 0$ and $s = 2$ into (4-37), we obtain the two equations:

$$
10 = 8A + 2B
$$

(4-39)
and
\[ \frac{10}{(4s+10)} = \frac{4}{4} \cdot \frac{2}{10} \]
or
\[ 4s + 8 = 5 \]
\[ 5s + 20 = 5 \]
which give the same values of \( A \) and \( B \) as before.

3. Heaviside's Expansion Theorem. The coefficients in (4-37) may be found by yet another technique, known as Heaviside's expansion theorem. This procedure can be justified, in the case of (4-37), by noting that multiplication of both sides by \( s + 2 \) gives us the equation
\[ \frac{10}{s + 2} = \frac{A}{s + 2} + \frac{B}{s - 8} \]
(4-49)

Since (4-49) holds for all values of \( s \), we can obtain an equation for \( A \) by letting \( s = -2 \), which eliminates the second term on the right-hand side. The result is
\[ A = \frac{10}{(-10)} = \frac{1}{10} \]
so before. Similarly, \( B \) can be found by multiplying both sides of (4-37) by \( s + 8 \) and setting \( s = -8 \). This results in
\[ \frac{10}{s + 8} = \frac{A}{s + 8} + \frac{B}{s - 2} \]

or \( B = \frac{10}{(-16)} = \frac{1}{16} \), as before.

We have just seen how to expand a rational function of \( s \), which contains only simple factors in the denominator, in partial fractions. That is, the procedures used in Example 4-8 can be used as long as the factors in the denominator are different and are not raised to a power. Furthermore, the degree of the numerator must be less than the degree of the denominator; that is, \( A \) must be a proper rational function of \( s \). If not, long division is used to obtain a proper rational function.

We now wish to consider examples involving more complicated factors than simple ones in the denominator.

EXAMPLE 4-9
As our next example, suppose that the input to the circuit shown in Figure 4-3 is \( x(t) = -0.5 \cos(2t) + 2 \sin(3t) \), with Laplace transform
\[ \mathcal{L}[x(t)] = -\frac{0.5}{s^2 + 1} + \frac{2}{s^3 + 1} \]

and that \( v(t) = -2 - t \). All other parameters are assumed to be the same as in Example 4-7. Then the Laplace transform of the output is
\[ \mathcal{L}[y(t)] = \frac{15s^2 + 25s + 20}{(s^2 + 1)(s^3 + 3s^2 + 3s + 2)} \]

Again, we have several approaches at our disposal for obtaining the partial-fraction expansion of (4-41). For example, the factor \( s + 1 \) in the denominator can be factored as \( (s + 1)(s + 1)(s + 1) \) and Heaviside's expansion theorem can be used as in part (3) of Example 4-8, using such a procedure, we would represent (4-41) as
\[ \mathcal{L}[y(t)] = \frac{A_1}{s + 1} + \frac{A_2}{s + 1} + \frac{A_3}{s^3 + 3s^2 + 3s + 2} \]

Using Heaviside's expansion formula, \( A_1 \) and \( A_2 \) are found to be
\[ A_1 = \frac{15}{(s + 1)} \]
\[ A_2 = \frac{25}{(s + 1)(s + 1)} \]
and
\[ A_3 = \frac{10}{s^3 + 3s^2 + 3s + 2} \]

respectively. \( A_1 \) and \( A_2 \) could be found similarly. However, by examining the first two terms in (4-42) further, we can simplify things to some extent. Putting the first two terms over a common denominator, we obtain
\[ A_1 \frac{1}{s + 1} + A_2 \frac{1}{s + 1} = \frac{A_1 + A_2}{s + 1} \]

The numerator of (4-43) must be a real function of \( s \). The only way that this can be true is for \( A_3 = A_4 \), so that \( A_1 + A_2 = 2 \). Hence, \( A_4 = 2 \) and \( A_3 = -2 \) too. Thus (4-42) can be written as
\[ \mathcal{L}[y(t)] = 2 \left[ \frac{A_1 + A_2}{s + 1} + \frac{A_3}{s^3 + 3s^2 + 3s + 2} \right] \]

This is a much simpler result. The unknown constants \( A_1 \), \( A_2 \), and \( A_3 \) in the final form on the right-hand side can be found in two ways.
EXAMPLE 4-10

In the example we consider the case of repeated linear factors. Let the forcing function for the circuit shown in Figure 4-3 be 

\[ y(t) = \frac{1}{s + 1} \exp(-2t) \cos(3t), \]

for which the Laplace transform is

\[ Y(s) = \frac{1}{s + 1} \exp(-2t) \cos(3t). \]

Assume that \( \varphi(0^+) = 0 \) and that the other parameters are the same as in Example 4-8. Thus

\[ Y(s) = \frac{1}{s + 1} \exp(-2t) \cos(3t), \]

The partial-fraction expression of this rational function must be of the form

\[ Y(s) = \frac{A_1}{s + 1} + \frac{A_2}{(s + 1)(s + 2)}, \]

where \( A_1 \) and \( A_2 \) can be found using the Heaviside technique. We obtain

\[ A_1 = (s + 1)Y(s), \]

\[ A_2 = (s + 1)Y(s), \]

and

\[ A_1 = (s + 1)Y(s), \]

The same approach will not work for \( A_1 \) because multiplication by \( (s + 1)^2 \) requires only one of the \( (s + 1) \) factors in the denominator of (4-50). Substitution of \( s = -1 \) then gives an undefined result for \( A_1 \). We could use the procedure employed in the preceding example and subtract \( A_1(1/s + 1) \) and \( A_2/s + 2 \) from this result. However, it is easier to note that

\[ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) Y(s) = \frac{d^2 A_1}{ds^2} + \frac{d A_2}{ds}, \]

or

\[ A_1 = \frac{\partial}{\partial t} Y(s), \]

and

\[ A_2 = \frac{\partial}{\partial t} Y(s), \]

Thus

\[ Y(t) = \frac{20}{9} \left( \frac{1}{s + 1} \exp(-2t) \cos(3t) + \frac{1}{s + 2} \exp(-2t) \cos(3t) \right). \]
Using Table 4-3, we find the inverse Laplace transform to be
\[ \mathcal{L}^{-1}\{e^{-2s} \} = \frac{1}{s-2} - \frac{1}{s-1} \]

**EXAMPLE 4-11**

We consider next the generalization of the method for finding the partial-fraction expansion for repeated linear factors developed in Example 4-10. Suppose that the transfer function for the circuit of Figure 4-3 is now \( \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-2s}y(t)\} \), with all other circuit parameters the same as in Example 4-10. Then
\[ \mathcal{L}\{e^{-2s}\} = \frac{1}{s+2} \]

and
\[ \mathcal{L}\{1\} = \frac{10}{s+2} \]

The partial-fraction expansion of \( \mathcal{L}\{y(t)\} \) is of the form
\[ \mathcal{L}\{y(t)\} = \frac{A_2}{s+2} + \frac{A_1}{s+1} + \frac{A_0}{s} \]

Using Heaviside's expansion formula, we obtain:

\[ A_0 = 10 \]
\[ A_1 = -10 \]
\[ A_2 = 0 \]

Using the differentiation technique of Example 4-10, we find that
\[ \frac{d}{ds}\mathcal{L}\{e^{-2s}y(t)\} = \frac{10}{(s+1)(s+2)} \]

\[ = \frac{8}{(s+1)(s+2)} \]

To find \( A_3 \), we note that multiplication of both sides of (4-53) by \( s + 2 \) gives
\[ \mathcal{L}\{e^{-2s}y(t)\} = \frac{10(s+2)}{s+2} + (s+2)(A_2 + 2A_1 + A_0) \]

Differentiation twice with respect to \( s \) gives
\[ \frac{d^2}{ds^2}(s + 2)\mathcal{L}\{y(t)\} = \frac{d^2}{ds^2}(s + 2)(s + 1)(s) \]

Setting \( s = -1 \), we obtain
\[ 3A_0 = \frac{d^2}{ds^2}(16) = \frac{160}{(s+1)(s+2)} \]
\[ = \frac{d}{ds}(s+1) \]

Thus \( A_0 = \frac{160}{16} = 10 \)

Using Table 4-3, we find \( \mathcal{L}\{1\} \) to be
\[ \mathcal{L}\{1\} = \frac{1}{s-2} - \frac{1}{s-1} = \frac{1}{s} \text{ for } s > 0 \]
EXAMPLE 4-12
Complex-conjugate roots. Consider the rational function of \( z \):
\[
Y(z) = \frac{2z^2 + 6z + 5}{z^2 + 2z + 2 - 1} = \frac{2z^2 + 6z + 5}{z + 1} \tag{4-60}
\]

The quadratic factor in the denominator can be factored as
\[
(z + 1)^2 = (z + 1)(z + 1) = (2 + i\sqrt{3})(2 - i\sqrt{3})
\]

and Hurwitz's criterion then used as in Example 4-8 since all factors are distinct. However, it is easier to keep both of the complex-conjugate factors expressed by (4-61) together and expand (4-60) as
\[
Y(z) = \frac{A}{z + 1} + \frac{B}{z + 1} \tag{4-62}
\]

This allows the inverse Laplace transform to be found easily with the help of pairs 3, 6, and 7 in Table 4-3.

As before, we may calculate \( A \) using Hurwitz's theorem:
\[
A = \frac{1}{1} \left[ 1 \right] \tag{4-63}
\]

The coefficients \( B \) and \( C \) may be obtained by substituting specific values of \( z \) in (4-62), or by placing both terms on the right-hand side over a common denominator and equating coefficients of like powers of \( z \) in the numerator. Of course, one could subtract \( \frac{1}{1} (z + 1) \) from \( Y(z) \) to produce the second term on the right-hand side of (4-62). We will use the first technique. First, let \( z = 0 \) in (4-62), we obtain
\[
0 = A + \frac{C}{2} \tag{4-64}
\]

or
\[
C = 2(0) - A = -A \tag{4-65}
\]

To find \( B \), we multiply both sides of (4-62) by \( z \) and let \( z = \infty \). This gives
\[
\lim_{z \to \infty} zY(z) = A + B \tag{4-66}
\]

The inverse Laplace transform is then
\[
Y(s) = \frac{1}{s + 1} + \frac{1}{s + 1} \tag{4-67}
\]

EXAMPLE 4-13 Repeated Quadratic Factors
Consider the rational function
\[
Y(z) = \frac{z^2 + 2z + 2}{z^2 + 2z + 2} \tag{4-68}
\]

The second factor in the denominator could be expanded as \( (z + j\beta)(z - j\beta) \) and (4-15) used to treat the repeated root. However, it is more convenient to keep the complex-conjugate factors in the denominator together and expand (4-68) as
\[
Y(z) = \frac{z + 1}{z + 1} \tag{4-69}
\]

Using Hurwitz's technique, we find \( A \) to be
\[
A = \frac{2}{2} = 1 \tag{4-70}
\]

We can obtain the remaining coefficients by placing the right-hand side of (4-69) over a common denominator and equating the resulting numerator to the numerator of (4-69). This results in the identity
\[
A + B = 2 \tag{4-71}
\]

Multiplying the factors on the left-hand side together and collecting like powers of \( z \), we obtain
\[
A + B = (1 + j\beta)(1 - j\beta) = 2 \tag{4-72}
\]

\[
A + B = 2 \tag{4-73}
\]

\[
A = 2 \tag{4-74}
\]

\[
B = 0 \tag{4-75}
\]

Thus (4-67) becomes
\[
Y(s) = \frac{1}{s + 1} \tag{4-76}
\]

Application of transforms pairs 3, 6, and 7 in Table 4-3 then yields
\[
Y(t) = e^{-t} \sin t + \sin t \tag{4-77}
\]
The coefficients of like powers of \( s \) on either side of (4-64) must be equal since (4-68) is an identity. This yields in the equations

\[
A_s + B_s = 1 \quad (4-69a)
\]
\[
C_s - 2A_s = 5 \quad (4-69b)
\]
\[
2B_s + B_s + 2B_s = 12 \quad (4-69c)
\]
\[
A_s + 2C_s + 2C_s = 15 \quad (4-69d)
\]

Since \( A_s = 1 \), the last of these equations results in

\[B_s = 1 - A_s = 0\]

This immediately gives

\[C_s = 5 \quad (4-69c)
\]
from (4-69b). Thus (4-69c) and (4-69d) become

\[2B_s + 12 = 12 \quad \text{or} \quad B_s = 0\]

and

\[1 + 2C_s = 15 \quad \text{or} \quad C_s = 2\]

respectively. The partial-fraction expansion for \( X(s) \), as given by (4-67), is

\[
X(s) = \frac{5}{s} + \frac{3}{s^2 + 1} + \frac{2}{s^2 + 4} \quad (4-70)
\]

This can be inverse Laplace transformed by using the theorem and transform pairs given earlier. The first two terms are immediately inverse transformed by using pairs 3 and 1, respectively, of Table 4-3. The last term can be handled by using the convolution theorem on pair 5 of Table 4-3. Thus

\[
X(t) = 5te^t + \sin t + 3\cos t + 2\sin 2t \quad (4-71)
\]

THE INVERSE LAPLACE TRANSFORM AND ITS USE IN OBTAINING INVERSE LAPLACE TRANSFORMS

In the introductory discussion on the Laplace transform given at the beginning of the chapter, we derived the inverse Laplace (4-3) in a logical way by making use of the inverse Fourier transform. In that derivation it was assumed constant.

It is much more useful to allow \( s \) to be a complex variable where both \( a \) and \( \omega \) may vary. The inversion integral in this case takes the form

\[
F(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st} \, ds
\]

where \( c > 0, \) and \( c, \omega \), with \( a \), is in the region of convergence of \( F(s) \).

To evaluate (4-71) we may make use of the residue theorem of the theory of complex variables.\(^3\) The residue theorem permits functions of a complex variable \( F(s) \) that possess a finite number of isolated singular points in the complex plane. Near each singular point, say \( s = s_0 \), the function can be represented as a series of the form (referred to as a Laurent series)

\[
F(s) = \sum \frac{a_n}{s - s_0} ^ n
\]

The \( a_n \)'s are coefficients which can be found by various means. The coefficient \( a_{-1} \) is referred to as the residue of \( F(s) \) at the singular point \( s = s_0 \).

With this introduction, we may now state the residue theorem:

**Residue Theorem**

Let \( C \) be a closed path in the complex plane and on which a function \( F(s) \) is analytic except for the isolated singular points \( s_1, s_2, \ldots, s_n \), with

\[\text{[Note: Appendix C-4 is a short survey of some pertinent definitions and theorems of complex analysis and their use in the Laplace transform theory.]}\]

\[^{3}\] Note that other theorems of complex variables, such as one theorem on the infinitesimal of a function of a complex variable, have singular points without residue. For example, in the inversion theorem the function is differentiable at a point (see Appendix C-4).
residues \( K_1, K_2, \ldots, K_n \) respectively. Then

\[
\int_0^\infty f(t) e^{-st} \, dt = 2\pi i \sum K_k e^{s_i} \quad (s_i > 0) \tag{4.74}
\]

where \( \int_0^\infty f(t) e^{-st} \, dt \) is understood to mean the line integral in a counterclockwise sense along \( C \).

How do we make use of the residue theorem to evaluate the inverse Laplace transform of a function of a complex variable, say \( \mathcal{L}^{-1} \{ \mathcal{X}(s) \} \)? To use the residue theorem to evaluate the integral of \( \mathcal{X}(e^{s}) \) along a line parallel to the imaginary axis \( s = \sigma + is \), we can apply the residue theorem by choosing a semi-circular path in the complex plane as shown in Figure 4.4. Using \( \mathcal{L}^{-1} \{ \mathcal{X}(s) \} \) in the residue theorem (4.4.24) we obtain

\[
\frac{1}{2\pi i} \int_{C_{o}} \mathcal{X}(e^{s}) \, ds = K_1 + K_2 + \cdots + K_n \tag{4.75}
\]

where \( K_1, K_2, \ldots, K_n \) are the residues of \( \mathcal{X}(e^{s}) \), and \( C_{o} \) is a contour enclosing the closed path consisting of the straight paths \( C_1 \) and \( C_2 \). Further, noting that the integral over the closed path \( C_1 + C_2 \) can be written as the sum of the integrals over the separate paths \( C_1 \) and \( C_2 \), but that the integral over \( C_2 \) is an integral between the limits \( e^{-\beta} \) to \( e^{\beta} + i \beta \), we obtain

\[
\frac{1}{2\pi i} \int_{C_{o}} \mathcal{X}(e^{s}) \, ds = \frac{1}{2\pi i} \int_{C_{1}} \mathcal{X}(e^{s}) \, ds = \sum_{n} K_n \tag{4.76}
\]

FIGURE 4.4. Closed contour in the complex plane that is appropriate for evaluating the inverse Laplace transform integral.

It can be shown that for \( s > 3 \) the second integral on the left-hand side contributes nothing to the result provided that \( \mathcal{X}(\sigma) = 0 \) uniformly as \( \sigma \to -\infty \). (By uniformly, it is meant that the limit is approached at the same rate for all angles of a within the range defined by \( C_2 \).) For example, if \( \mathcal{X}(s) \) is the ratio of two polynomials, it is sufficient for the degree of the denominator to exceed that of the numerator by one or more. Given this restriction on \( \mathcal{X}(s) \), it follows that the left-hand side of (4.74) yields the inverse Laplace transform, as defined by (4.73), in the limit as \( s \to 0 \). That is, for \( s > 0 \),

\[
x(t) = \sum \text{residues of } \mathcal{X}(e^{s}) \text{ at the finite singularities of } \mathcal{X}(s) \tag{4.77}
\]

EXAMPLE 4.14

As an example, consider the function \( x \) given by (4.25) which was inverse transformed in Example 4.4. The function \( \mathcal{X}(s) \) has two singularities which are first order poles, given by

\[
s_1 = -3 + \beta
\]

Using the Heaviside expansion technique, we write

\[
\mathcal{X}(s) = \frac{s + \beta}{(s + \beta)} = \frac{A_1}{s + 3 + \beta} - \frac{A_2}{s + 3 - \beta}
\]

where

\[
A_1 = (s + 3 - \beta)(L) \quad \text{and} \quad A_2 = 0
\]

and

\[
A_1 = (s + 3 - \beta)(L) \quad \text{and} \quad A_2 = 0
\]

We require the residues of

\[
\mathcal{X}(e^{s}) = \frac{s + \beta}{(s + \beta)} = \frac{A_1 e^{s}}{s + 3 + \beta} - \frac{A_2 e^{s}}{s + 3 - \beta}
\]

at the poles of \( \mathcal{X}(s) \). That is, we want the coefficient \( K_1 \), of the expansion

\[
A_1 e^{s} \quad \text{at } \quad s = -3 + \beta
\]

and the coefficient \( K_2 \), of the expansion

\[
A_2 e^{s} \quad \text{at } \quad s = -3 - \beta
\]

Now since \( e^{s} = \sum_{n=0}^{\infty} s^n/n! \), it is apparent that \( s^n \) has no singularities in the

4.5 THE INVERSION INTEGRAL AND ITS USE IN OBTAINING INVERSE LAPLACE TRANSFORMS
Note: The plane because the opposite-power terms are present in its Laurent series expansion. It follows, therefore, that by applying the Heaviside technique. That is,

\[ K = (1 + 3 - J) \int \frac{e^{-s}}{s^3} \, ds \]

and

\[ K' = (1 + 3 - J) \int \frac{e^{-s}}{s^3} \, ds \]

From (4.47) we obtain, for \( r > 0 \),

\[ g(r) = \frac{1}{2} \left( \frac{1}{2} - J \right) e^{\frac{i}{2} \theta} \sin \left( \frac{\theta}{2} \right) \]

which is the same result as obtained in Example 4.4.

Generally, the inversion integral would not be used for a problem such as this. It is easier to use partial-fraction expansion and a table of Laplace transform pairs.

### 4.6 THE DOUBLE-SIDED LAPLACE TRANSFORM

In the discussion leading to (4.41) it was assumed that the signal \( s(t) \) was zero for \( t < 0 \). This resulted in the single-sided Laplace transform which was partially adequate for our purposes. Removing this restriction on \( s(t) \) and allowing the same reasoning that resulted in (4.41) we obtain the double-sided Laplace transform of a signal \( s(t) \), which is

\[ S(s) = \int_{-\infty}^{\infty} s(t) e^{-st} \, dt \quad (4.76) \]

The inversion integral remains the same as before with an added restriction on choosing the path of integration which will be discussed shortly. The integral

\[ \int_{-\infty}^{\infty} g(s) \, ds \]

converges absolutely if

\[ \int_{-\infty}^{\infty} |s(t)| \, dt < \infty \]

But

\[ \int_{-\infty}^{\infty} |s(t)| \, dt < \infty \]

and the double-sided Laplace transform integral therefore converges absolutely if

\[ \int_{-\infty}^{\infty} |s(t)| \, dt < \infty \]

A sufficient condition on \( s(t) \) for (4.76) to hold is that there exist a real positive number \( A \) so that for some real \( t \) and \( s \), \( |s(t)| \) is bounded by

\[ |s(t)| \leq Ae^{-At} \]

for \( t > 0 \)

\[ |s(t)| \leq Ae^{At} \]

for \( t < 0 \)

Then (4.76) is absolutely convergent for \( \varepsilon < \sigma < B \). When using contour integration to evaluate the inverse double-sided Laplace transform, the path of integration is chosen within the convergence strip and closed by a semicircular arc to the left for \( \sigma > 0 \) and to the right for \( x > 0 \). To show (4.81), we break the integral (4.76) up into two parts as

\[ S(x) = \int_{-\infty}^{0} s(t) e^{-st} \, dt + \int_{0}^{\infty} s(t) e^{-st} \, dt \]

and use (4.80) to bound \( X(s) \) by

\[ X(s) = A e^{-At} \frac{1}{s - A} + \int_{0}^{\infty} s(t) e^{-st} \, dt \]

The first integral will converge for \( \Re(s) < a \) and lower limit yields zero when substituted. and the second integral will converge for \( \Re(s) > c \) (the upper limit yields zero when substituted). It follows that the singularities of \( X(s) \) due to the pole-zero cancellation at \( s = c \) lie to the left of the line \( \Re(s) = c \), while the singularities of \( X(s) \) due to the pole-zero cancellation at \( s = A \) lie to the right of the line \( \Re(s) = A \). In contrast to the single-sided Laplace transform, the region of convergence must be specified together with a two-sided Laplace transform in order to identify the corresponding inverse transform uniquely. To illustrate this, consider the two-sided Laplace transform of the signal

\[ s(t) = e^{at} \]

The Laplace transformation applications to systems analysis.
and
\[ s(t) = -e^{\alpha t}(1 - t) \]
Both have the two-sided Laplace transform \(1/(s - 1)\). However, their regions of convergence are different. In particular:
\[ X_1(s) = \frac{1}{s - 1} \quad \text{for} \quad s > 1 \]
and
\[ X_2(s) = \frac{1}{s} \quad \text{for} \quad s < 1 \]
where \(X_1(s)\) and \(X_2(s)\) are the Laplace transforms of \(x_1(t)\) and \(x_2(t)\), respectively.

We may obtain the two-sided Laplace transform of a signal through one of the single-sided Laplace transform by separating the signal into its positive and negative-time components. In particular, assume that \(x(t)\) is defined for \(-\infty < t < \infty\). We may represent it as:
\[ x(t) = x(t_0 + t) \quad (4.84) \]
where \(x(t) = x(t_0 + t)\) \( \quad (4.85a) \)
and
\[ x(t) = x(t_0 - t) \quad (4.85b) \]
The two-sided Laplace transform of \(x(t)\) is
\[ X_{2s}(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \]
\[ = \int_{-\infty}^{t_0} x(t)e^{-st} \, dt + \int_{t_0}^{\infty} x(t)e^{-st} \, dt \quad (4.85c) \]
where \(X_{2s}(s)\) denotes the double-sided Laplace transform. Changing variables is the first integral to \( \tau = -t \), we obtain
\[ X_{2s}(s) = \int_{-\infty}^{t_0} x(-\tau)e^{-s(-\tau)} \, d\tau + \int_{t_0}^{\infty} x(t)e^{-st} \, dt \quad (4.86) \]
where
\[ X_{2s}(s) = X(s) + X(\bar{s}) \quad (4.88a) \]
and
\[ X_{2s}(s) = X(s) - X(\bar{s}) \quad (4.88b) \]
The reason \(X_{2s}(s)\) denotes the single-sided Laplace transform. The signal \(x(t)\) is, of course, the reflection (mirror image) of the negative-time portion of \(x(t)\) about \(t = 0\). Equation (4.87) tells us to take the single-sided Laplace transform of this signal, replace \(s\) by \(\bar{s}\) in the result, and add these two functions of \(s\) to the single-sided Laplace transform of \(x(t)\), which is the positive-time portion of \(x(\bar{s})\).

**EXAMPLE 4-15**

Find the two-sided Laplace transform of
\[ x(t) = e^{2t}(-t + 1) \]
**Solution:** The signals \(x_1(t)\) and \(x_2(t)\) are
\[ x_1(t) = e^{2t} \quad (4.89) \]
and
\[ x_2(t) = -e^{2t} \quad (4.89b) \]
Their single-sided Laplace transforms are:
\[ X_1(s) = \frac{1}{s + 2} \quad (4.89a) \]
and
\[ X_2(s) = \frac{1}{s + 2} \quad (4.89b) \]
Replacing \(s\) by \(\bar{s}\) in \(X_1(s)\), we obtain
\[ X_2(s) = \frac{1}{s + 2} \quad (4.89b) \]
\[ \text{Re}(-s) < 0 \quad \text{or} \quad s > 2 \]
Therefore, from (4.87), we obtain
\[ X(s) = \frac{1}{s + 3} \quad (4.87) \]
\[ s + 3 = 0 \quad s = -3 \quad -3 < s < 2 \]
**EXAMPLE 4-16**

Find the inverse double-sided Laplace transform of
\[ X(s) = \frac{1}{s + 3} \quad (4.87) \]
\[ -3 < s < 2 \]
**Solution:** The partial-fraction expansion of \(X(s)\) is
\[ X(s) = \frac{1}{s + 3} \quad (4.87) \]
\[ s + 3 = 0 \quad s = -3 \quad -3 < s < 2 \]
Since the pole at \(s = -3\) lies to the left of the convergence region, the term \(1/(s + 3)\) contributes to the Laplace transform of the positive-time portion of \(x(t)\). Similarly, since the pole at \(s = -1\) lies to the right of the convergence region, the term \(-1/(s + 1)\) contributes to the Laplace transform of the negative-time portion of \(X(\bar{s})\). Using (4.87), (4.89a), and (4.89b), we obtain
\[ x(t) = \frac{1}{s + 3} \quad (4.87) \]
\[ s + 3 = 0 \quad s = -3 \quad -3 < s < 2 \]
\[ s + 3 = 0 \quad s = 3 \quad 3 < s < 2 \]
and

\[ x(t+1) = 2x(t) - 2x(t-1) - \alpha x(t) \]

Putting these results together we obtain

\[ d(t) = 2e^{\alpha t} - e^{\alpha t} 2e^{-\alpha t} \]

**SUMMARY**

In this chapter the basic tool for the analysis of lumped, fixed, linear systems has been introduced, namely the Laplace transform. It is advantageous in that it provides both the transient and forced responses of the system. To employ it in system analysis, one proceeds by transforming the governing differential equations of the system into the complex frequency, or s domain. The Laplace-transformed output in response of the system can then be solved for algebraically. Inversion to the time domain for a large class of the problems encountered is accomplished by using partial-fraction expansion of the Laplace transform of the system response and using a table of Laplace transforms pairs. An alternative procedure for obtaining the Laplace transformed response is to use a Laplace-transformed equivalent circuit. Analysis proceeds then in a systematic fashion to using phasors and impedances in sinusoidal steady-state analysis. We consider this approach to system analysis in Chapter 5.

**FURTHER READING**


A revision of the second edition published in 1960. An excellent treatment of complex variables and applications offered at the senior- or junior-year graduate level.


The Laplace transforms are introduced in Chapter 4 and complex variables are treated in Chapter 10 of this reference.


Chapter 5 contains an introduction to the Laplace transforms which begins with double-sided and specializes in the single-sided Laplace transform.


The treatment of Laplace transforms given in Chapter 6 of this book is similar in scope and level to the coverage in this text.


The Laplace transform is covered in Chapter 8 of this reference. Both the inversion integral and inversion by partial-fraction expansion are considered.


The scope of this book is large and extensive. An excellent treatment of complex-variable theory and inversion of the Laplace transform by contour integration. The applications to network analysis given in the text are at a level somewhat above those considered here.


A very readable and short introduction to Laplace transforms is given in Chapter 1. Other very readable, elementary treatments of Laplace transforms is given in Chapter 16 in about the same level as Van Valkenburg.


This excellent text in circuit theory is mentioned here because of its completeness. The Laplace transform is treated in Chapter 13 with s = 1"lower level employed. It is at a slightly more difficult level than the preceding two references.

**PROBLEMS**

**SECTIONS 4-1 and 4-2**

4-1. Obtain the Laplace transform of the following signals:

(a) \( (1 - e^{-3t})u(t) \)

(b) \( (e^t - e^{3t})u(t) \)

(c) \( (e^t - e^{-t})u(t) \)

(d) \( (e^t - e^{-t})u(t) \)

**SECTION 4-3**

4-2. Obtain the Laplace transforms of the following signals:

(a) \( \cos(200\pi t) \)

(b) \( \sin(200\pi t) \)

(c) \( \sqrt{2} \cos(200\pi t - \pi/4) \).

(Note: Relax this signal to those given in parts (a) and (b).)

4-3. Obtain the Laplace transform of the triangular signal \( x(t) = t^2(t) \) by using the differentiation theorem, the time delay theorem, and expressing \( dx/dt \) in terms of unit step.

4-4. Solve the following differential equation by means of the Laplace transforms:

(a) \( \frac{d^2x(t)}{dt^2} - 2\frac{dx(t)}{dt} + x(t) = 0 \)

(b) \( \frac{dx(t)}{dt} + 3x(t) = 2(t) \)

(c) \( \frac{dx(t)}{dt} + 3x(t) = 0 \)

4-5. Solve for the current in the circuit shown in the switch changes as indicated by the arrow as \( \alpha = 0 \). Assume that the switch has been in position 1 since \( t = \infty \).