(b) Generalise the result of part (a) to n stages and show that

\[ R(z) = (1 + A(z))^n \]

where \( A(z) \) is the transfer function of an RC stage repeated as above.

(c) Show that the impulse response of the n-stage RC-coupled amplifier is

\[ h[n] = \left[ \begin{array}{c} h_1^n[n] \\ \vdots \\ h_n^n[n] \end{array} \right] \]

Using the results of Problems 6.07, show that the effective delay and duration for the impulse response of the n-stage RC-coupled amplifier are given by

\[ h_1^n \approx RC \quad \text{and} \quad h_2^n \approx 2RC, \quad \text{respectively.} \]

**State-Variable Techniques**

6.1 **INTRODUCTION**

So far, the techniques of system analysis we have discussed have been applied primarily to systems with only a single input and a single output. Although such systems often serve as smaller parts of a network, they are rarely sufficient for modeling the overall system. More engineering systems of importance involve several inputs and several outputs. Although such a system can be modeled as an interconnection of single-input, single-output systems, it is not always advantageous to do so. As we shall see, the state-variable method provides a convenient formulation for these systems. It allows us to handle systems with multiple inputs and outputs within precisely the same mathematical framework that we will use for single-input, single-output systems.

The state-variable formulation has other advantages as well. In previous discussions, we have assumed that there is a direct relationship between the system input and output. No information concerning the internal behavior of the system was included in the formulation. On the other hand, the state-variable formulation allows us to determine the internal behavior of the system easily while still giving the input-output information we desire. Furthermore, the state-variable formulation is often the most efficient form from which the complete of
STATE-VARIABLE CONCEPTS

There are several concepts which, if understood, will make our further discussions both easier and more meaningful. The most fundamental concept is that of state of a system. Whenever we speak of the state of a system, we refer to the state in a compact form which we believe contains more information than its components. We may say that the state of a system at time $t_0$ is a sequence of all the variables necessary to completely define the condition of the system at time $t_0$. If we allow determination of all system variables at any time $t$, when inputs to time are specified, also the state of a system at time $t = t_0$, if it is a linear, time-variant electric network, for example, knowledge of all capacitor voltages and inductor currents uniquely specifies the network condition at any particular time. Similarly, if we have knowledge of the input alone so as to find the condition of the network at any future time. This is for the example of the capacitor voltages and inductor currents of a system at time $t_0$.

The state of a system at any is simply the set of states, at time $t_0$, of an approximately chosen set of variables. These variables are called the state vari-

able. For the electric network example above, the state variables are the capacitor voltages and inductor currents. The number of state variables is equal to the order of the system and it can be derived from the number of energy storage elements. The value of any state variables is not unique. In fact, there are generally infinite state variables from a single equation. The equation $a_1 \cdot x(t) + \cdots + a_n \cdot x(t) = b_1 \cdot y(t) + \cdots + b_m \cdot y(t)$

The above equation is in state form. In this equation $x(t)$ is a state variable which is defined as an output of some system at time $t$. The state vector $x(t)$ contains $n$ elements, each element is defined by $x_k(t)$, where $x_k(t)$ is the component at time $t$. The state vector is a vector whose $i$th component is the state variable. Thus at any time $t$ the state vector defines a point in the state space. The state of the system at $t = 0$ is the initial state of the system. As $t$ increases, the states change, and a set of points will be defined. This set of points, the locus of the tip of the state vector as time increases,

The concepts and terminology of this section are used throughout the remainder of this chapter. The reader may find it helpful to refer to this section from time to time.

FORM OF THE STATE EQUATIONS

An alternative linear differential equation with one output and $n$ states can always be represented by a linear differential equation with $n$ inputs as in the following example:

$$x_1 = a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n + b_1 \cdot u_1 + \cdots + b_n \cdot u_n$$

$$x_2 = a_{21} \cdot x_1 + \cdots + a_{2n} \cdot x_n + b_2 \cdot u_1 + \cdots + b_n \cdot u_n$$

$$\vdots$$

$$x_n = a_{n1} \cdot x_1 + \cdots + a_{nn} \cdot x_n + b_1 \cdot u_1 + \cdots + b_n \cdot u_n$$

where the dot indicates time differentiation $\dot{x}_i = dx_i/dt$. The $a_{ij}$ are the system inputs, and the $x_i$ are the system outputs. These $x_i$ are called the state variables. The $b_i$ are often called the input variables. It is important to distinguish the state variables from the input variables. Together, the $x_i$ and $b_i$ variables constitute a state equation model for the system. The state model consists of a set of $n$ first-order state equations. To find the state model of a system, one must determine the $a_{ij}$, $b_i$ and $d_i$ functions of time. However, the solution of each state equation is a $n$-th order differential equation which is generally difficult and must therefore always be carried out with the aid of a computer. For this reason we restrict our attention to state equation systems, one for which all the coefficients are constant. In this case we can solve solutions without too much difficulty, as we shall see in the following section.

As the point is a first-order system with two inputs. Thus, we have the following two equations. Each, one of the state variables $x_1$ and $x_2$ with two inputs $u_1$ and $u_2$.

$$x_1 = \frac{dx_1}{dt} = a_{11} \cdot x_1 + a_{12} \cdot x_2 + b_1 \cdot u_1 + b_2 \cdot u_2$$

$$x_2 = \frac{dx_2}{dt} = a_{21} \cdot x_1 + a_{22} \cdot x_2 + b_1 \cdot u_1 + b_2 \cdot u_2$$

where $a_{ij}$ are the system inputs, and the $x_i$ are the system outputs. We shall see how this is done in the next section. The next obvious advantage of state space representation is that it allows us to easily handle the case of multiple inputs and multiple outputs. This is something that can be done only with a comprehensive time-domain representation, which flow through the approach when dealing with model validation without interfering in it.
considerable difficulty using the other analysis techniques we have discussed. Of course, the single-input single-output system is included in a special case (i.e., k = 0 and h = 0). For this situation a 2 = 0.

Thus far, the model is given in the time domain, and it is straightforward to obtain a simulation diagram for the equation. This is extremely useful if we wish to use computer simulation methods to study the system. Furthermore, an extremely compact matrix notation can be used for the state model. Using the laws of matrix algebra, the above system can be rewritten in matrix form:

\[ x(t) = Ax(t) + Bu(t) \]

\[ y(t) = Cx(t) + Du(t) \]

Thus, we can derive the simulation diagram for the state equation. For brevity, we consider a two-input, one-output second-order system given by:

\[ x_1 = x_2 + 2x_1 + u_1 \]

\[ x_2 = x_1 + 2x_2 + u_2 \]

\[ y = x_1 + x_2 \]

\[ (S3) \]

It is evident that if we know the states, we could find the output, for the inputs and all coefficients are known below. Therefore, consider only the first two equations. If we have \( x_1 \) and \( x_2 \), we can find \( x_1 \) and \( x_2 \) by simple integration. Hence \( x_1 \) and \( x_2 \) should be the inputs of two integrators. The corresponding integrator inputs are \( u_1 \) and \( u_2 \). This leaves only the problem of obtaining \( x_1 \) and \( x_2 \) for use as inputs to the integrators. In fact, it is already specified by the state equations. \( x_1 \) and \( x_2 \) are exactly the appropriate linear combinations of \( x_1, x_2, u_1, u_2 \) specified by the first two equations (S3). Once we do this, we simply use the inputs \( u_1 \) and \( u_2 \) for the integrator outputs \( x_1 \) and \( x_2 \) to form the system output. The completed diagram is shown in Figure 5.1. These findings are exactly as expected. Since the state variables are defined as \( x_1 \) and \( x_2 \), the two integrators are connected to the two summing points to reflect the mathematical statement of the state model. Of course, the diagram becomes simpler and more simplified, but this is to be expected—no more equations.

An examination of the simulation diagram reveals the significance of the various elements of the state model. To begin, essentially specify the internal system structure and behavior, even in the absence of inputs, but with initial states assigned to the system. Just as these initial states specify the starting value of each state variable, which internal behavior of the system. Finally, the applied inputs specify the internal behavior of the system. In other words, the states are transformed in a manner such that the system evolves to its final state. The states are then monitored and/or measured to provide the system behavior. The states are then monitored and/or measured to provide the desired system behavior. The states are then monitored and/or measured to provide the desired system behavior. This can be obtained directly from the simulation diagram. It is perhaps worth noting that the very equations can be obtained directly from a simulation diagram. All that is required is to write an expression for the output of each state variable in terms of its inputs. In Figure 5.1 the inputs are the common output and input model \( u_1 \) and \( u_2 \). Summing these equations of (S3). Similar operations at the other components valid, but this is to be expected—no more equations.

**Figure 5-1:** Simulation diagram of the system of (S3).

\[ x_1 = x_2 + 2x_1 + u_1 \]

\[ x_2 = x_1 + 2x_2 + u_2 \]

\[ y = x_1 + x_2 \]

\[ (S3) \]
and matrices

\[
\begin{align*}
A &= \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}, & B &= \begin{bmatrix}
    b_{1} \\
    b_{2} \\
    \vdots \\
    b_{n}
\end{bmatrix}, & C &= \begin{bmatrix}
    c_{1} \\
    c_{2} \\
    \vdots \\
    c_{n}
\end{bmatrix}, & D &= \begin{bmatrix}
    d_{1} & d_{2} & \cdots & d_{m}
\end{bmatrix}
\end{align*}
\]

Thus, using the usual rules of matrix algebra, it is obvious that the state model of its 1 and 1) may be written compactly as

\[x = Ax + Bu
\]

(6.6)

These equations may also be indicated schematically by a block diagram as shown in Figure 8.2. The study here is investigation of the multiple cause signal flow path of the appropriate derivative. The block diagram matrices multiplication of the appropriate vector and matrices. The block representing the integrator in fact contains a sum of appropriate components specified by the A and B matrices. Of course, to obtain a movable transfer function, we must know all the elements of the matrices, but at least this block diagram represents the essential elements of the system and the signal flow paths. A similar diagram may be made regarding the state equations themselves. Before we can obtain numerical solutions in any particular case, we must locate all elements of the block matrices. B, C, and D. However, as in most problems, its generality of construction is the major portion of the work involved in actually implementing the state model.

Another advantage of the state model is that the matrix elements provide a convenient means of describing these various subsystems or subsystems only in general properties of the system, without regard to particular numerical values. We explore this advantage in the following section in which solutions of the state equations.

8.4 TIME-DOMAIN SOLUTION OF THE STATE EQUATIONS

For the purpose of developing the solution of the state equations, let us consider the matrix equation

\[x(t) = Ax(t) + Bu(t)
\]

(6.7)

We will assume that the matrices are non-singular. A non-singularity of the solution

\[x(t) = e^{At}u(t)
\]

where

\[e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}
\]

is the so-called matrix exponential. The matrix exponential is defined by

\[e^{At} = \exp(At)
\]

and satisfies

\[\exp(A + B) = \exp(A)\exp(B)
\]

for all matrices A and B. The matrix exponential is important in the solution of linear systems of the form

\[\dot{x} = Ax + Bu
\]

where x is the state vector, u is the input vector, and A and B are constant matrices. The matrix exponential can be used to solve initial value problems of the form

\[\dot{x} = Ax + Bu, \quad x(0) = x_0
\]

where x is the state vector at time t = 0, u is the input vector, and A and B are constant matrices. The matrix exponential is defined by

\[e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}
\]

and satisfies

\[\exp(A + B) = \exp(A)\exp(B)
\]

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\[\dot{x} = Ax + Bu, \quad x(0) = x_0
\]

where x is the state vector at time t = 0, u is the input vector, and A and B are constant matrices. The matrix exponential is defined by

\[e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}
\]

and satisfies

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for all matrices A and B. The matrix exponential is important in the solution of linear systems of the form

\[\dot{x} = Ax + Bu
\]

where x is the state vector, u is the input vector, and A and B are constant matrices. The matrix exponential can be used to solve initial value problems of the form

\[\dot{x} = Ax + Bu, \quad x(0) = x_0
\]
where \( k \) is any non-negative integer. Upon reverting to the state equations at time \( t = t + x \Delta t \), we have:

\[
W_{x,t+\Delta t} = (x+1)W_{x,t} - W_{x-1,t} = \Delta t \frac{\partial W}{\partial x}
\]

or

\[
W_{x,t+\Delta t} = (x+1)W_{x,t} + W_{x-1,t} = \Delta t \frac{\partial W}{\partial x} + 
\]

Equation (6-11) provides the desired recursive relations. If \( W_{x,t} \) is known, it can

be used to find the values of \( W_{x,t+\Delta t} \) and \( W_{x-1,t+\Delta t} \). This equation is of the form of a difference equation. We shall study such equations in some detail in the following chapter.

As an example of a difference equation, we shall take the following example:

\[
x = A + Bw, \quad w = \mu
\]

recalling that \( \mu \) is known, \( x \) can be easily found. However, the approach familiar to us from differential equations, \( \mu \) should, at least, be a solution of the second-order, and

\[
x = \lambda + \mu, \quad w = \mu
\]

\[
\partial W/\partial x = \mu
\]

\[
W_{x,t+\Delta t} = (x+1)W_{x,t} + W_{x-1,t} = \Delta t \frac{\partial W}{\partial x} + 
\]

where \( \lambda \) is the solution to the homogeneous equation and \( \mu \) is a particular solution. The homogeneous solution must satisfy the initial condition.

The homogeneous equation is given by

\[
\lambda = \Delta t \frac{\partial W}{\partial x}
\]

By analogy to the scalar equation:

\[
x = \mu, \quad w = \mu
\]

\[
\partial W/\partial x = \mu
\]

we might guess the solution of (6-10) to be

\[
x/\mu = e^{\Delta t \mu}
\]

Immediately a question arises as to the meaning of the matrix exponential. We shall have more to say about it later, but for now let us assume that it can be

\[
ed^\mu = I
\]

where \( I \) is an \( n \times n \) identity matrix. Then

\[
\mu = \lambda^\mu = \mu H \cdot \mu = A^\mu
\]

and the recursive solution satisfies the homogeneous equation. Furthermore, in view of (6-17),

\[
x/\mu = e^{\Delta t \mu} = y
\]

Now (6-11) gives the complete solution of the system. Before discussing the particular solution we digress to note some properties of the matrix exponential. These properties may be divided in terms of its infinite series expansion as follows:

\[
ed^\mu = I + \mu A + \frac{1}{2!} \mu^2 A^2 + \cdots
\]

Clearly, \( e^{\mu} = \mu + \cdots + \mu^n \). The matrices on the right-hand side are not positive definite. The matrices on the right-hand side are zero on the

\[
(e^{\mu})^{-1} = e^{-\mu}
\]

Now let us complete the solution of the state equations by determining the particular solution. We adopt the method of variation of parameters and assume

\[
\lambda^\mu = e^{\Delta t \mu}
\]

when \( x(t) \) is a scalar function to be found. Differentiating (5-20) and setting that \( x(t) \) enter the original equation:

\[
x(t) = \lambda x(t) = e^{\Delta t \mu} \cdot x(t)
\]

\[
\frac{d}{dt} x(t) = \mu \cdot x(t)
\]

\[
\frac{d}{dt} x(t) = \lambda x(t)
\]

\[
\frac{d}{dt} x(t) = \mu x(t)
\]

\[
x(t) = e^{\Delta t \mu} x(t)
\]

\[
\mu x(t) = e^{\Delta t \mu} x(t)
\]

\[
\mu x(t) = e^{\Delta t \mu} x(t)
\]

\[
\mu x(t) = e^{\Delta t \mu} x(t)
\]

\[
x(t) = e^{\Delta t \mu} x(t)
\]

\[
\mu x(t) = e^{\Delta t \mu} x(t)
\]
Now, the particular solution must be such that
\[ x_p(t) = 0 \]
since the homogeneous solution already satisfies the initial conditions. It can be shown that \( x_p'''' = 0 \) is nonsingular; therefore, from (6-25) we must have \( x_p(t) = 0 \). Thus
\[ x(t) = x_h(t) + x_p(t) \]
Using this result in (6-25) then yields the particular solution as
\[ x_p(t) = x_h(t) + x_p(t) \]
or
\[ x(t) = x_h(t) + x_p(t) \]
The complete solution is obtained by adding the homogeneous solution and the particular solution as follows:
\[ x(t) = x_h(t) + x_p(t) \]

The origin ofhomogeneous solution for a system whose state-space description is given can be determined by solving the characteristic equation. Using the roots of the characteristic equation, we obtain the homogeneous solution as
\[ x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t} \]
The trace is the sum of the state transition matrix. Once the input is specified, the state transition matrix explicitly describes the manner in which the system state changes from its initial value \( x(0) = x_0 \) to its value at any time \( t \). After we examine an alternative method of solving the state equations, the following section, we shall describe some techniques of finding the state transition matrix, the next, we may write the complete system response as
\[ y(t) = (C + [0 \ldots 0]) x(t) = (C + [0 \ldots 0]) (B + [0 \ldots 0]) x_0 \]

**6-6 FREQUENCY-DOMAIN SOLUTION OF THE STATE EQUATIONS**

An alternative method of solving the state equations makes use of the Laplace transform technique discussed in Chapter 5. This method is quite attractive from an analytical point of view. Again we are concerned primarily with
\[ i = A x + B u \]
\[ x(t) = x_0 \]
For convenience, we shall assume that \( x(0) = 0 \). Laplace transforming (6-27) yields
\[ sX(s) - x(0) = AX(s) + BU(s) \]
Rearranging yields
\[ (s - A)X(s) = BU(s) \]
or
\[ X(s) = (s - A)^{-1}BU(s) \]
When we take the inverse transform we must then apply the rules given by our previous analysis, with \( x(0) = 0 \). From the first term of (6-29) it is evident that
\[ Z = (s - A)^{-1} \]
where \( F(s) \) is the Laplace transform of the state transition matrix. The second term of (6-29) must yield the integral term of (6-27), which is consistent with the convolution property of a product of Laplace transforms,
\[ Z \cdot (s - A)^{-1} \]

Therefore, the time-domain solution is exactly as we have written in the preceding section. In the frequency domain, the response of the system is given by
\[ Y(s) = C \cdot (s - A)^{-1} \cdot (B + [0 \ldots 0]) \]
\[ = C \cdot (s - A)^{-1} \cdot B + \ldots + C \cdot (s - A)^{-1} B \]

Equation (6-31) is simply Laplace transforms of (6-26).

For complex input, block-oriented systems we defined the transfer function as the ratio of the Laplace transform of the output to the Laplace transform of the input, with all initial conditions at zero. We extend this concept to the frequency domain as shown in the next section.
Hence we may write

$$H_{ij} = R_{i}(k_{i})$$  \( \text{(3.30)} \)

Since there are \( m \) inputs and \( n \) outputs, \( H_{i,j} \) is a matrix having elements \( H_{i,j} = 1, 2, \ldots, m \). The function \( R_{i}(k_{i}) \) is the transfer function between the \( i \)-th input and the \( j \)-th output.

We may also define an impulse response matrix \( B(t) \)

$$H(t) \approx 2^{\beta}(B(t))$$  \( \text{(3.37)} \)

Hence \( H(t) \) may be written as

$$H(t) = B(t)H(0) + \int_{0}^{t} \dot{B}(t)H(0)dt$$ \( \text{(3.36)} \)

where \( B(t) \) is an \( m \times n \) matrix of unit impulses. \( H(0) \) has the same dimensions as \( H(t) \) and its elements \( h_{i,j}(0) \) are

$$h_{i,j}(t) = \begin{cases} h_{i,j} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$ \( \text{(3.36)} \)

where \( h_{i,j} \) is equal to \( \delta_{i,j} \) when a unit impulse is applied at the \( j \)-th input.

The formal solution of the state equation is now complete. We have an expression that gives the system output for any input and any set of initial conditions. The matrix multiplications and the integral of \( B(t) \), which are handled separately, are straightforward. The integral may be evaluated either directly (as done for \( H(t) \)) or by employing partial fraction expansions. If we use a time-domain approach, it is clear that we must know the state transition matrix \( B(t) = B_{i,j}(t) \). We now turn to this problem.

### 6.5 FINDING THE STATE TRANSITION MATRIX

Two techniques for determining the state transition matrix of a given system have already been mentioned, although a specific identification is not given. These techniques include the use of the state method or of the Laplace transforms method described in Section 5.3. Recall that we defined the state transition matrix in terms of its infinite series

$$B(t) = B_{i,j} = \frac{A_{i}}{z} + \frac{A_{i}z}{z^{2}} + \frac{A_{i}z^{2}}{z^{3}} + \cdots$$ \( \text{(3.20)} \)

For a given \( A(z) \), \( B(t) \) may be evaluated to any desired degree of accuracy by performing the indicated matrix multiplications and summations. A simple example will illustrate the procedure.

#### EXAMPLE 6.1

Find the state transition matrix \( B(t) \) for the following transfer function:

$$A(z) = 1 - z$$

It is immediately found that

$$A(z) = 1 - z, \quad A(z)^{-1} = z - 1, \quad A^{*}(z) = z - 1, \quad 0 < \beta < 0$$

Thus,

$$A^{*}(z) = 1 - z, \quad 0 < \beta < 0, \quad A^{*}(z)^{-1} = z - 1$$

We immediately recognize the series expansions of \( 1 - z \) and \( z - 1 \):

$$A^{*}(z) = 1 - z$$

This approach may be satisfactory for many applications in which the impulse technique may be used to calculate a transfer function of a system. However, when a circuit contains many elements, it is usually necessary to employ a time-domain approach in order to obtain the solution. The following example should prove convincing.

#### EXAMPLE 6.2

Find the state transition matrix for the following transfer function:

$$A(z) = 0$$

We have

$$A^{*}(z) = 0$$

$$0 = 0$$

$$A^{*}(z) = 0$$

$$0 = 0$$

$$A^{*}(z) = 0$$

$$0 = 0$$

$$A^{*}(z) = 0$$

$$0 = 0$$

We conclude that the state transition matrix is

$$B(t) = 0$$

$$\text{Thus, } B(t) = 0$$

$$\text{The state transition matrix is } B(t) = 0.$$
and so on. Using this result in (5-43) yields

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \]

This expression for \( e^x \) can be shown to be the best first order of \( e^x \).

The Laplace transform of the generating function makes possible the use of the initial condition:

\[ \Phi(s) = \frac{1}{s^2 + 1} \]

We simply take \( \Phi(s) = A_2 \), take the inverse Fourier transform, and then take the inverse Laplace transform to get \( \Phi(t) = e^t \). Partial fraction expansion methods are usually helpful in taking the inverse transforms. For example, will discuss this method.

**Example 6.3**

Use the Laplace transform method to find the state transition matrix of

\[ A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \]

We have

\[ (sI - A)^{-1} = \begin{bmatrix} 5s & 4 \\ 5 & s \end{bmatrix} \]

Performing a partial fraction expansion of each term and inverse transforming yields:

\[ e^{at} = 2 \left( \frac{\Phi(s)}{s - \lambda} \right) = \frac{e^{\lambda t}}{1 + 2} \]

as per Example 6.2.

The Laplace transform method is quite convenient for analytical work since it yields a more compact form. However, it is not very useful for machine computation. There exist other techniques for finding \( e^x \) which are often easier.

One such technique involves finding a suitable coordinate transformation which, when applied to the system, converts this error into a constant form. Once this is done the state transition matrix is found by applying the inverse coordinate transformation. A fourth technique makes use of the Cauer canonical form.

The coefficients of the series expansion are found by solving a set of simultaneous algebraic equations. These results are based on the Cayley-Hamilton theorem (see Appendix A). The coefficients of the power series representation become functions of time. Therefore, it is possible to write

\[ \Phi(s) = e^s \]

When the eigenvalues of \( A \) are distinct it is straightforward to determine the coefficients \( n(r) \). Suppose that the eigenvalues, \( \lambda_1, \lambda_2, \ldots, \lambda_r \), are distinct. Then we use (6-43) with \( A \) replaced by each \( \lambda_i \) to obtain a set of \( n(r) \) equations:

\[ e^{\lambda_i t} = n(0) + n(1) \lambda_i t + n(2) \lambda_i^2 t^2 + \ldots + n(r) \lambda_i^r t^r \]

Equation (6-43) may be solved for the coefficients \( n(r) \), \( r = 0, 1, 2, \ldots \).

\[ e^{\lambda_i t} = n(0) + n(1) \lambda_i t + n(2) \lambda_i^2 t^2 + \ldots + n(r) \lambda_i^r t^r \]

Equation (6-43) may be solved for the coefficients \( n(r) \), \( r = 0, 1, 2, \ldots \).

\[ e^{\lambda_i t} = n(0) + n(1) \lambda_i t + n(2) \lambda_i^2 t^2 + \ldots + n(r) \lambda_i^r t^r \]
EXAMPLE 6.4

Use the Cayley-Hamilton approach to find the state transition matrix for

\[ A = \begin{bmatrix} 0 & 1 \\ -5 & -5 \end{bmatrix} \]

The characteristic equation is

\[ \det (A - \lambda I) = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0 \]

Clearly, the eigenvalues are \( \lambda_1 = -2 \), \( \lambda_2 = -3 \)

We have

\[ e^{\lambda t} = e^{\lambda_1 t}I + \lambda_1 tI \]

and from (6.31) we get

\[ e^{\lambda t} = e^{\lambda_1 t}I + \lambda_1 tI \]

Substituting, we obtain

\[ \lambda_1 I + \lambda_1 tI \Rightarrow \lambda_1 = -2 \]

It follows that

\[ \lambda_1 I + \lambda_1 tI \Rightarrow \lambda_1 = -3 \]

Then

\[ e^{\lambda t} = (e^{\lambda_1 t}I + \lambda_1 tI) \]

or

\[ e^{\lambda t} = \begin{bmatrix} e^{-2t} - 3e^{-3t} \\ -5e^{-2t} + 6e^{-3t} \end{bmatrix} \]

which agrees with the results of the preceding two examples.

If some or all of the eigenvalues are repeated, the method must be modified accordingly. Since this discussion is not meant to be exhaustive, we refer the interested reader to the references at the end of the chapter.

We are given the state transition matrix to find the state transition matrix of a system described by a state model. Therefore, we solve the system with two

EXAMPLE 6.5

Find the impulse response, given

\[ A = \begin{bmatrix} 0 & 3 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

where

\[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

and \( v(t) \) is a unit step input. This is a single-input, two-output system. From (6.36) we have the Laplace transform of the system as

\[ Y(s) = C(sI - A)^{-1}B + C(sI - A)^{-1}B(s) \]

First, we obtain \((sI - A)^{-1}\) in

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s+2} \\ \frac{3}{s+2} & \frac{3}{s+2} \end{bmatrix} \]

Then

\[ Y(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \]

Finally:

\[ Y(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ 1 \]

\[ 0 \]

\[ 0 \]

\[ 0 \]

\[ 0 \]
Now, canceling the $s + 1$ terms in the bottom row occurs and applying the partial fraction expansion yields

$$\begin{align*}
\text{Yin} &= \frac{1}{s + 1} \left( \frac{2}{s + 1} - \frac{1}{s + 1} \right) \\
&= \frac{1}{s + 1} - \frac{1}{s + 1}
\end{align*}$$

The first term in the initial condition response, while the second is the transient response. Combining the terms yields the complete response in its simplest form as

$$\text{Yin} = \frac{1}{s + 1} - \frac{1}{s + 1}$$

Now that we have explicitly found the $\text{Yin}$, we instead carried through all the realizations and our differences in terms of the transforms and the final state. Unless $\text{Yin}$ is explicitly solvable, this is usually the second procedure, or it leads to somewhat simpler functions which must be treated separately.

EXAMPLE 6-6

In this example we consider a system with two inputs and two outputs. Such systems are difficult to handle unless we use state-variable methods. For simplicity, we use a system similar to that in Example 6-3 and its initial conditions for ease. The system equations are assumed to be:

$$\begin{align*}
\dot{x}_1 &= -2x_1 + x_2 \\
\dot{x}_2 &= 2x_1 + 3x_2 \\
\text{Yout} &= 1x_1 + 1x_2
\end{align*}$$

We assume that $x_1(0) = 0$, $x_2(0) = 0$, and $\text{Yout}(0) = 0$. For $\text{Outline}$ (see page 270).

Using $-A^{-1}$ from Example 6-5, we obtain

$$\begin{align*}
\text{Yout} &= \left[ \begin{array}{c}
0 \\
0
\end{array} \right] + \left[ \begin{array}{c}
1 \\
-1
\end{array} \right] \\
&= \left[ \begin{array}{c}
0 \\
0
\end{array} \right] + \left[ \begin{array}{c}
1 \\
-1
\end{array} \right] \\
&= \left[ \begin{array}{c}
1 \\
-1
\end{array} \right]
\end{align*}$$

or

$$\begin{align*}
\text{Yout} &= \left[ \begin{array}{c}
1 \\
-1
\end{array} \right]
\end{align*}$$

Finally, we obtain

$$\begin{align*}
\text{Yout} &= \left[ \begin{array}{c}
1 \\
-1
\end{array} \right]
\end{align*}$$

We may write the state transition with the aid of the partial-fraction expansion to obtain

$$\begin{align*}
\text{Yin} &= \left[ \begin{array}{c}
1 \\
-1
\end{array} \right]
\end{align*}$$

Two comments are perhaps in order at this point. First, it would be difficult to write the equation of this example without the use of state-variable method. The reader should convince himself of this by drawing a block diagram of the equations and considering how it might be used to obtain $y_1$ and $y_2$. Second, although we have only treated a second-order system, the same techniques can be used on systems of any order. The algebra, of course, becomes tedious for higher-order systems.
STATE EQUATIONS FOR ELECTRICAL NETWORKS

We now consider the problem of obtaining state equations for systems. We consider only the special case of electrical networks here. However, by the use of suitable analogies, the technique can be extended to other classes of lumped-parameter systems (e.g., mechanical systems).

Earlier in this chapter we noted that a suitable set of state variables for electrical networks were the capacitive voltages and inductive currents. Physically, this choice is attractive because the capacitive voltages and inductive currents specify the stored energy. These are convenient choices from a mathematical point of view as well. We use the element relations for capacitors and inductors as

\[ \mathbf{L} \frac{d}{dt} \mathbf{v} + \mathbf{C} \mathbf{v} = \mathbf{E} \]

Note that the determinants of the chosen state variables appear in these expressions. If we use node voltages for the capacitor current and inductor voltage (in terms of the state variables and source voltages and in currents, then we can obtain equations for the node derivative of state variable \( \mathbf{n} \times \mathbf{n} \) linear combination of state variables and inputs.

This provides the form of the state equations. The source current and voltage variables are also expressed in terms of state variables and source quantities by writing suitable KCL and KVL equations. The output equation can also be obtained in this way. Let us summarize the method in the following algorithm and then consider some examples.

**Algorithm**

1. Express capacitive voltages and inductive currents as state variables.
2. For each capacitor, write a KCL equation expressing the capacitor currents (i.e., in terms of state variables, source quantities, and other currents as necessary).
3. For each inductor, write a KVL equation expressing the inductive voltages (i.e., in terms of state variables, source quantities, and other voltages as necessary).
4. Write other KCL, KVL, and element source equations as necessary to eliminate the "leftover" currents and voltages (as steps 2 and 3) in terms of the state variables, source quantities, and other currents or voltages, after dividing by \( \mathbf{C} \) or \( \mathbf{L} \) as appropriate, to obtain the state equations.

If we use node voltages for the capacitor current and inductor voltage (as in the algorithm), the state model is then

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

**EXAMPLE 6-7**

(Refer to the model for the network shown in Figure 6.3.)

![Figure 6.3](image)
EXAMPLE 5.6

Write state equations for the two-input, two-output circuit shown in Figure 5.6.4. The inputs are \( v_1 \) and \( v_2 \), and the outputs are \( v_4 \) and \( v_5 \).

Again we choose \( i_1 \) and \( i_2 \) as state variables. A KCL for the capacitor yields:

\[ \frac{v_1}{C_1} + \frac{v_2}{C_2} + \frac{v_4}{C_3} = i_1 \]

A KVL around the left-hand loop gives:

\[ v_1 - v_4 = \frac{1}{C_1} \int i_2 \, dt \]

Turning our attention to the inductor current equations, we see that we must eliminate \( i_2 \). We have:

\[ i_1 = R_1 i_2 \]

and writing a loop equation around the loop defined by \( v_2, R_2, R_3 \), and \( C \) yields:

\[ \frac{v_2}{R_2} + \frac{v_4 - v_5}{R_3} = -i_2 \]

Since:

\[ i_2 = \frac{v_1}{R_1} \]

we have:

\[ \frac{v_2}{R_2} + \frac{v_4 - v_5}{R_3} = \frac{v_1}{R_1} \]

and:

\[ \frac{v_2}{R_2} - \frac{v_4 - v_5}{R_3} = \frac{v_1}{R_1} \]

Thus:

\[ v_4 - v_5 = \frac{v_1}{R_1} \]

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We may put all this in matrix form as
\[
\begin{bmatrix}
\frac{1}{CRk} & \frac{1}{CRk} & 0 & \cdots & 0 \\
-\frac{1}{CRk} & \frac{1}{CRk} + \frac{1}{kR} & \frac{1}{kR} & \cdots & 0 \\
0 & \frac{1}{kR} & \frac{1}{kR} + \frac{1}{2R} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{mR} + \frac{1}{(m-1)R} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{Ck} \\
\frac{1}{kR} \\
\frac{1}{kR} + \frac{1}{2R} \\
\vdots \\
\frac{1}{mR} + \frac{1}{(m-1)R} \\
0
\end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{mR} + \frac{1}{(m-1)R} \\ \frac{1}{kR} + \frac{1}{2R} \\ \vdots \\ \frac{1}{kR} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{mR} + \frac{1}{(m-1)R} \\ \frac{1}{kR} + \frac{1}{2R} \\ \vdots \\ \frac{1}{kR} \\ 0 \end{bmatrix}
\]

While the second of our examples requires considerably more equations to be written in order to eliminate the unknown variables, both of the examples show that the algorithm may be applied to derive the state equations. Of course, more complicated circuits may be handled by the same method. The effort certainly increases.

8.8 STATE EQUATIONS FOR nTH-ORDER DIFFERENTIAL EQUATIONS

We have commented previously that a system which is represented by an nth-order ordinary differential equation can also be represented by a first-order differential equations in \( n \) by state equations. We close this chapter with a brief discussion of how this may be accomplished. Only one technique is presented here. The interested reader is referred to Ogata’s book for a discussion of other methods.

Let us suppose that a system is represented by the nth-order differential equation
\[
x'''' + 3x''' + 2x'' + x' + x = 0
\]
where \( x \) is the circuit input, \( x \) is the circuit output, and \( x' \) is \( \frac{dx}{dt} \). We wish to obtain the state model
\[
x = A \cdot x + b u
\]
\[
y = c x + d u
\]
Let us first consider the case where \( m = 0 \), that is, there are no derivatives of the input. Let us define state variables as
\[
x_1 = x, \quad x_2 = x', \quad x_3 = x'', \quad \ldots, \quad x_{n-1} = x^{(n-2)}
\]
Then
\[
x_1 = x
\]
\[
x_2 = x'
\]
\[
x_3 = x''
\]
\[
x_{n-1} = x^{(n-2)}
\]
\[
x_{n-1} = -ax_{n-1} - ax_{n-2} - \cdots - ax_2 + bx_1 + \mu
\]
\[
\mu = -ax_{n-1} - ax_{n-2} - \cdots - ax_2 + bx_1
\]
where the last equation is obtained from the differential equation (8-34) and the state variable definitions (8-33). We may write (8-47) as
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]
\[
\begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]
\[
\begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]
\[
\begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]
\[
\begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
y \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]

Of course, since the matrix is just \( I \), we have
\[
y = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} x_1 + \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \mu
\]

and \( d = 0 \). Equations (8-48) and (8-49) are the desired state model.

If the original differential equation contains derivatives of the input, the method must be modified somewhat. We may no longer choose state variables simply as the outputs and its derivatives. It is still possible, however, to choose state variables...
variables such that the equations look quite similar to (6-46). Let us consider the state variable case (i.e., $x_1$ and $y_1$) since we know what happens. Suppose that we choose the state variables as in (6-46). The system equations are the same as (6-47) except for the last one, which is

$$\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}y_1 + a_{14}y_2 + b_1y_3 + \bar{y} \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}y_1 + a_{24}y_2 + b_2y_3 + \bar{y} \\
\dot{y}_1 &= x_1 \\
\dot{y}_2 &= x_2
\end{align*}$$

However, the last term is not valid for state systems. Three different choices of state variables must be made. Let us choose:

- $x_1 = y_1$
- $x_2 = y_2$
- $x_3 = y_3$

Then

$$\begin{align*}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}y_1 + a_{14}y_2 + b_1y_3 + \bar{y} \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}y_1 + a_{24}y_2 + b_2y_3 + \bar{y} \\
\dot{x}_3 &= \dot{y}_1 \\
\dot{x}_4 &= \dot{y}_2
\end{align*}$$

That is, the state equations are

$$\dot{x} = Ax + Bu + \bar{y}$$

where $x = [x_1, x_2, y_1, y_2]^T$, $u = [y_3]^T$, $A$, $B$, and $C$ are defined as before.

Further Reading


Problems

**Section 5.3**

5.1. Construct a state variable diagram for the system described by

$$\begin{align*}
x_1 &= 2x_1 + 3x_2 + 3u \\
\dot{x}_2 &= -x_1 + 2u + x \\
y &= x_1 + 2y_2 + 4u
\end{align*}$$

Note that only the vector $b$ is changed from (6-46). The output equation is identical to (6-49).

For $n > 1$ this approach can be generalized to yield state equations whose $A$ matrix remains in the form of (5-50). Other forms are also possible. These details are beyond the scope of this text.
4.2. Construct a simulation diagram for the system whose state equations are
\[
\begin{align*}
x_1' &= -1 \ x_2 + y = 0 \\
x_2' &= -3 \ x_2 + y = 0
\end{align*}
\]
\[
y = (1 \ 3 \ -2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

4.3. Write state equations in matrix form which describe the simulation diagram.

\[
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

SECTION 6.4 AND 6.5

6.4. Use the series expansion method to compute \( e^{At} \)
\[
A = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}
\]

6.5. Use the series expansion method to compute \( e^{At} \)
\[
A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}
\]

6.6. Show that if \( x = \lambda x \) a matrix \( A \) is given by
\[
A = \lambda \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} e^{\lambda t} & 0 & \cdots & 0 \\ 0 & e^{\lambda t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda t} \end{pmatrix}
\]

6.7. For the \( A \) of Problem 6.6, find \( x(t) \) if \( x(0) \) is a unit step function and \( x(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

6.8. In 6.13 we gave the recursive equation for computing the approximate solution to the state equation. Let the system be described by
\[
\begin{align*}
x_1' &= -1 \ x_1 + \delta \ x_2 + \epsilon \ x_1 \\
x_2' &= -3 \ x_2 + \delta \ x_1 + \epsilon \ x_2
\end{align*}
\]
(a) With \( \delta = 1 \), use 30.13 to compute \( x(t) \)
(b) Repeat part (a) with \( \delta = 0 \)
(c) Compare the actual value of \( x(t) \) and compare with parts (a) and (b).

SECTION 6.5 AND 6.6

6.9. Use the Laplace transform method to find \( e^{At} \) for the \( A \) given in Problem 6.5.

6.10. Use the Laplace transform method to find \( e^{At} \)
\[
A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}
\]

\[
\begin{pmatrix} e^{0 \ t} & 0 \\ 0 & e^{-t} \end{pmatrix}
\]
6.11. Use the Cayley-Hamilton approach to find \( y(t) \) if
\[
\begin{bmatrix}
0 & 2 \\
-1 & 4
\end{bmatrix}
\]
6.12. Use the Laplace transform approach to find \( y(t) \) for the system given by
\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} = C \begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} + \begin{bmatrix}
y_0(t) \\
y_0(t)
\end{bmatrix}
\]
where \( y_0(t) = \frac{x(t)}{t} \) and \( x(t) \) is an input.

6.13. Find the transfer function matrix for the system
\[
\begin{bmatrix}
y_1(s) \\
y_2(s)
\end{bmatrix} = C \begin{bmatrix}
y_1(s) \\
y_2(s)
\end{bmatrix} + \begin{bmatrix}
y_0(s) \\
y_0(s)
\end{bmatrix}
\]

SECTION 6.4
6.14. Obtain a state model for the network shown.

6.15. Obtain a state model for the network shown.

SECTION 6.5
6.16. Obtain a state model for the network shown.

6.17. Obtain a state model for:
(a) \( y + 3x = 0 \)
(b) \( y + 5x = 3y + 3x \)
(c) \( y + 3x = 3y + 10y = 3y 

6.18. Obtain a state model for
\( \dot{x} + 2x + y = 3x + 4y \)