\[ H(s) = s^{-1} \text{L}(H)(s) \]  
\[ E(s) = s^{-1} \text{L}(E)(s) \]  
(3.83)
(3.84)

The significance of using \( s^{-1} \) is that it represents pure integrative in the time domain. Now, a signal flow graph using Eqs. (3.79), (3.80), (3.83), and (3.84) is constructed as shown in Fig. 3.23(c). Notice that in this signal flow graph the Laplace transform variable appears only in the form of \( s^{-1} \). Therefore, this signal flow graph may be used as a basis for analog or digital computer solution of the problem. Signal flow graphs in this form are defined in Chapter 4 as the state diagrams.  

3.10 General Gain Formula for Signal Flow Graphs

Given a signal flow graph or a block diagram, it is usually a tedious task to solve for its input-output relationships by analytical means. Fortunately, there is a general gain formula available which allows the determination of the input-output relationship of a signal flow graph by mere inspection. The general gain formula is

\[ M = \frac{\gamma_m}{\gamma_n} = \frac{N \cdot M_1 \cdot \Delta}{\Delta} \]  
(3.85)

where

- \( M \) = gain between \( \gamma_n \) and \( \gamma_m \)
- \( \gamma_m \) = output node variable
- \( \gamma_n \) = input node variable
- \( N \) = total number of forward paths
- \( M_1 \) = gain of the \( k \)th forward path
- \( \Delta \) = gain product of the \( m \)th possible combination of \( r \) non-touching loops

or

\[ \Delta = 1 - \sum P_{r1} + \sum P_{r2} - \sum P_{r3} + \ldots \]  
(3.86)

\[ P_{rn} = \text{gain product of all possible combinations of} \ r \ \text{non-touching loops} \]  

\[ \Delta_k = \text{the} \ \Delta \ \text{for that part of the signal flow graph which is non-touching with the} \ k \ \text{th forward path} \]  

This general formula may seem formidable to use at first glance. However, the only complicated term in the gain formula is \( \Delta \); but in practice, systems having a large number of non-touching loops are rare. An error that is frequently made with regard to the gain formula is the condition under which it is valid. It must be emphasized that the gain formula can be applied only between an input node and an output node.

*Two parts of a signal flow graph are said to be non-touching if they do not share a common node.
Mason’s Gain Rule is a technique for finding an overall transfer function. It is helpful when trying to simplify complex systems. The purpose of using Mason’s is the same as that of Block Reduction. However, Mason’s is guaranteed to yield a concise result via a direct procedure, where as the process of Block Reduction can meander. Mason’s method was particularly helpful before the advent of modern computers, and tools such as MatLab which can also be used to find the overall transfer function of a complex system (and then perform subsequent analysis).

The Dorf text presents Mason’s Gain Rule applied to signal flow diagrams. This presentation is for block diagrams only. A derivation of Mason’s Gain Rule is beyond the scope of the course. As such, just the procedure will be presented here. The Dorf text does present a specific example and draws parallel between Mason’s and the solution of a set of linear equations.

Mason’s procedure refers to portions of a block diagram in terms of ‘paths’ and ‘loops’. Loops begin and end at the same point, and are described in terms of the concatenation of all the transfer function blocks encountered. This is the ‘loop gain’ and also includes any negations associated with a summer. Hence there is one loop in the above system, \( L_1 = -G_1G_2H_1H_2 \). A path runs from the input \( C(s) \) to the output \( R(s) \) and is without any loops. Paths are also described by the concatenation of blocks. In the above system there is one path, described as \( G_1G_2 \).

Another criterion used in the evaluation of Mason’s has to do with the notion of ‘touching’ loops and paths. Touching loops contain a common branch (or summer or block) element. A loop touches a path if they share a common block or summer.

Mason’s Gain Rule states \( T = \frac{\sum_k P_k \Delta_k}{\Delta} \) where,

\[
\Delta = 1 - \text{(sum of all different loop gains)} \\
\quad + \text{(sum of products of all pairs of loop gains, for non-touching loops)} \\
\quad - \text{(sum of products of all triples of loop gains, for non-touching loops)} \\
\quad + \ldots
\]

\( P_k \) = \( k^{th} \) path from input to output.

\( \Delta_k \) = The quantity \( \Delta \), but with all loops touching the \( k^{th} \) path, \( P_k \), removed.

This is best illustrated via examples...
Example 1a. Find \( T(s) = \frac{R(s)}{C(s)} \), both algebraically and via Mason’s Gain Rule

\[
\begin{align*}
\text{Working algebraically,} \\
T &= \frac{G1}{1 + G1H1} \frac{G2}{1 + G2H2} = \frac{G1G2}{1 + G1H1 + G2H2 + G1G2H1H2}
\end{align*}
\]

Via Mason’s first note there is one path from \( C(s) \) to \( R(s) \):
\[ P1 = G1 \ G2 \]

There are two loops:
\[ L1 = -G1 \ H1 \]
\[ L2 = -G2 \ H2 \]

Loops L1 and L2 are not touching. Finding \( \Delta \),
\[ \Delta = 1 - (L1 + L2) + L1 \ L2 = 1 + G1H1 + G2H2 + G1G2H1H2 \]

To find \( \Delta_1 \), loops touching path \( P1 \) are removed from \( \Delta \). Hence \( \Delta_1 = 1 \).

So, Mason’s technique yields:
\[ T = \frac{P1 \ \Delta_1}{1 - (L1 + L2) + (L1L2)} = \frac{G1G2}{1 - (-G1H1 - G2H2) + G1H1G2H2} \]

Which is the same result as with the algebraic method.

1b) Similar to 1a, with the two loops now touching.

\[
\begin{align*}
\text{Working algebraically, first define a signal } D & \text{ that flows out of the central node. The} \\
\text{signal } D & \text{ flows along the } -H1 \text{ path and along the } G2 \text{ path. This can be reduced to:} \\
T &= \frac{G1G2}{1 + G1H1 + G2H2}
\end{align*}
\]

Which is the same as the result from Mason’s with the \( L1L2 \) term dropped due to the touching loops.
Example 2. Find $T(s) = \frac{R(s)}{C(s)}$, Algebraically and via Mason’s Gain Rule

There are two paths from $C(s)$ to $R(s)$: 
$$P_1 = G_1 \quad \text{and} \quad P_2 = G_2$$

There are two loops:
$$L_1 = -G_1 H_1$$
$$L_2 = -G_2 H_1$$

Loops $L_1$ and $L_2$ are touching. Finding $\Delta$,
$$\Delta = 1 - (L_1 + L_2) = 1 + G_1 H_1 + G_2 H_1$$

To find $\Delta_1$, loops touching path $P_1$ are removed from $\Delta$. $\Delta_2$ is found similarly. Hence
$$\Delta_1 = 1 \quad \text{and} \quad \Delta_2 = 1$$

So Mason’s technique yields:
$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{1 - (L_1 + L_2)} = \frac{G_1 + G_2}{1 + G_1 H_1 + G_2 H_1}$$

Working algebraically, define $G = G_1 + G_2$, then
$$T = \frac{G}{1 + GH} = \frac{G_1 + G_2}{1 + (G_1 + G_2)H}$$